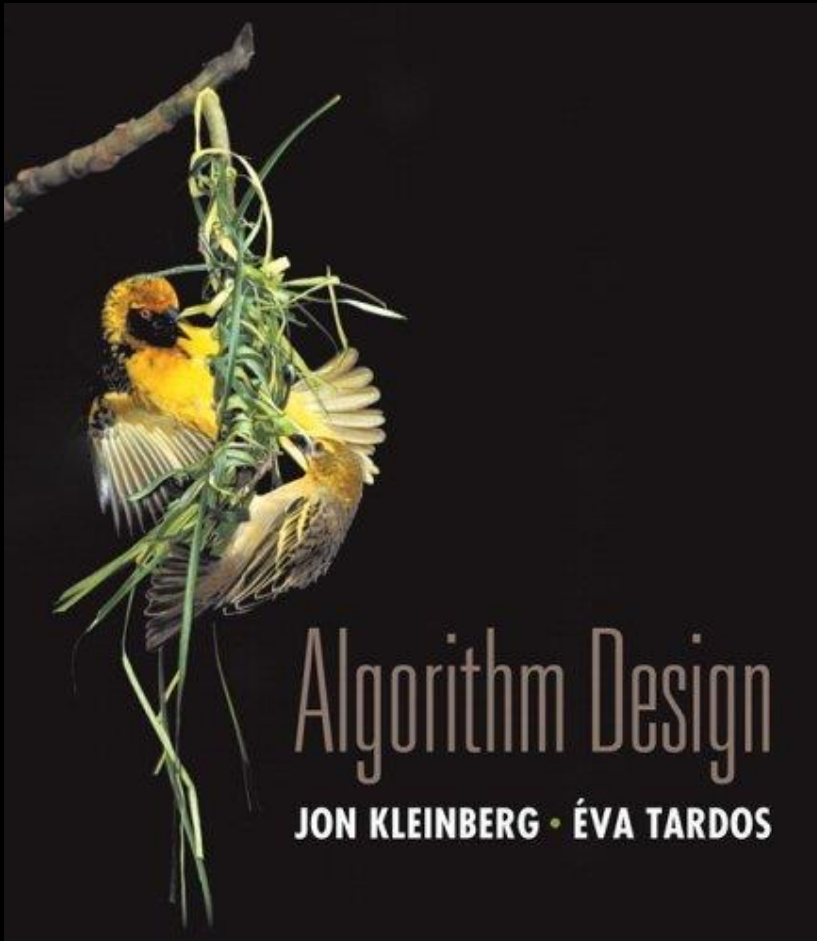


# Chapter 8

## NP and Computational Intractability



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# Algorithm Design Patterns and Anti-Patterns

## Algorithm design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Duality.
- **Reductions.**
- Local search.
- Randomization.

## Ex.

$O(n \log n)$  interval scheduling.

$O(n \log n)$  FFT.

$O(n^2)$  edit distance.

$O(n^3)$  bipartite matching.

## Algorithm design anti-patterns.

- **NP-completeness.**
- PSPACE-completeness.
- Undecidability.

$O(n^k)$  algorithm unlikely.

$O(n^k)$  certification algorithm unlikely.

No algorithm possible.

## 8.1 Polynomial-Time Reductions

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# Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [Cobham 1964, Edmonds 1965, Rabin 1966]  
Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover
Primality testing	Factoring

# Classify Problems

**Desiderata.** Classify problems according to those that can be solved in polynomial-time and those that cannot.

**Provably requires exponential-time.**

- Given a Turing machine, does it halt in at most  $k$  steps?
- Given a board position in an  $n$ -by- $n$  generalization of chess, can black guarantee a win?

**Frustrating news.** Huge number of fundamental problems have defied classification for decades.

**This chapter.** Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one **really hard** problem.

# Polynomial-Time Reduction

**Desiderata'**. Suppose we could solve  $X$  in polynomial-time. What else could we solve in polynomial time?

don't confuse with reduces from

**Reduction.** Problem  $X$  **polynomially reduces to** problem  $Y$  if arbitrary instances of problem  $X$  can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem  $Y$ .



**Notation.**  $X \leq_p Y$ .

computational model supplemented by special piece of hardware that solves instances of  $Y$  in a single step

**Remarks.**

- We pay for time to write down instances sent to black box  $\Rightarrow$  instances of  $Y$  must be of polynomial size.
- Note: Cook reducibility.



in contrast to Karp reductions

# Polynomial-Time Reduction

**Purpose.** Classify problems according to **relative** difficulty.

**Design algorithms.** If  $X \leq_p Y$  and  $Y$  can be solved in polynomial-time, then  $X$  **can** also be solved in polynomial time.

**Establish intractability.** If  $X \leq_p Y$  and  $X$  cannot be solved in polynomial-time, then  $Y$  **cannot** be solved in polynomial time.

**Establish equivalence.** If  $X \leq_p Y$  and  $Y \leq_p X$ , we use notation  $X \equiv_p Y$ .

↑  
up to cost of reduction

# Reduction By Simple Equivalence

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Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

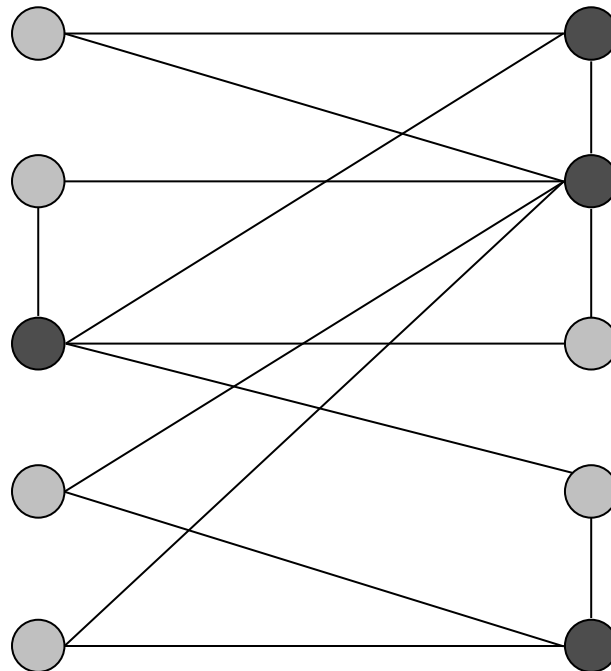


# Independent Set

**INDEPENDENT SET:** Given a graph  $G = (V, E)$  and an integer  $k$ , is there a subset of vertices  $S \subseteq V$  such that  $|S| \geq k$ , and for each edge at most one of its endpoints is in  $S$ ?

**Ex.** Is there an independent set of size  $\geq 6$ ? Yes.

**Ex.** Is there an independent set of size  $\geq 7$ ? No.



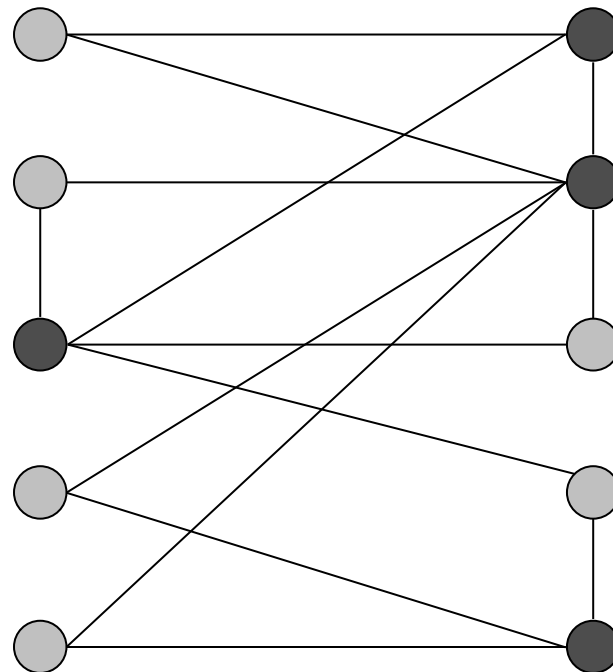
● independent set

# Vertex Cover

**VERTEX COVER:** Given a graph  $G = (V, E)$  and an integer  $k$ , is there a subset of vertices  $S \subseteq V$  such that  $|S| \leq k$ , and for each edge, at least one of its endpoints is in  $S$ ?

**Ex.** Is there a vertex cover of size  $\leq 4$ ? Yes.

**Ex.** Is there a vertex cover of size  $\leq 3$ ? No.

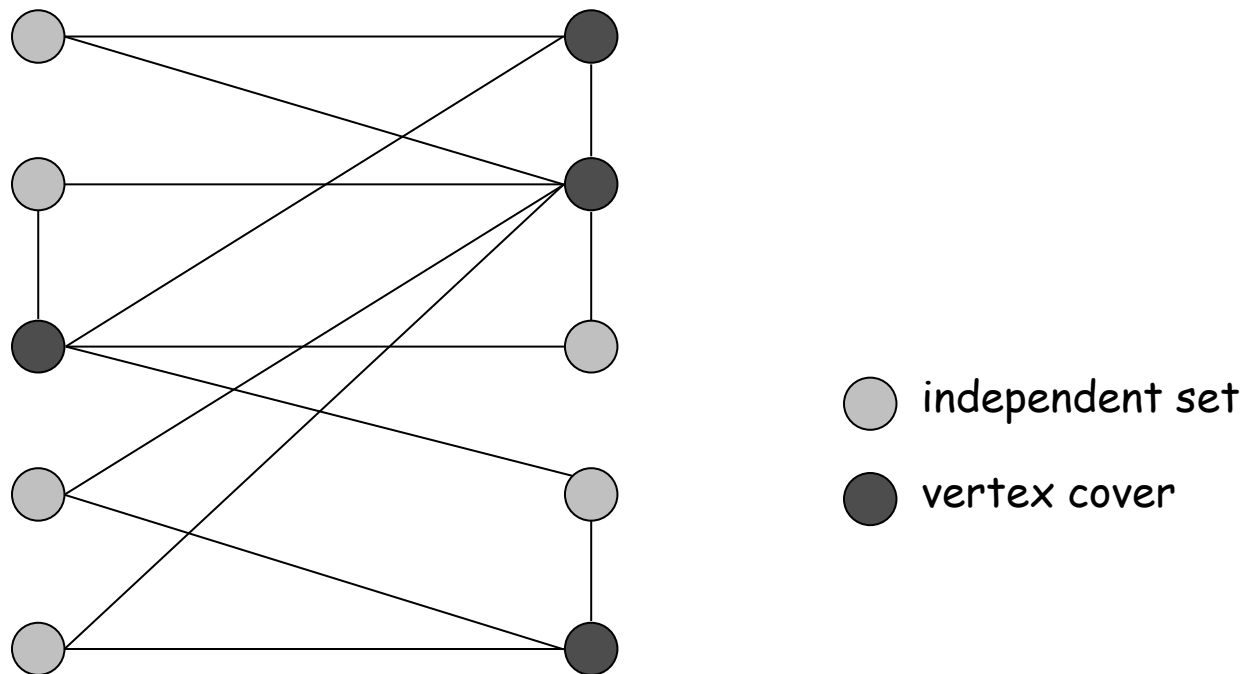


● vertex cover

# Vertex Cover and Independent Set

**Claim.** VERTEX-COVER  $\equiv_p$  INDEPENDENT-SET.

**Pf.** We show  $S$  is an independent set iff  $V - S$  is a vertex cover.



# Vertex Cover and Independent Set

**Claim.** VERTEX-COVER  $\equiv_p$  INDEPENDENT-SET.

**Pf.** We show  $S$  is an independent set iff  $V - S$  is a vertex cover.

$\Rightarrow$

- Let  $S$  be any independent set.
- Consider an arbitrary edge  $(u, v)$ .
- $S$  independent  $\Rightarrow u \notin S$  or  $v \notin S \Rightarrow u \in V - S$  or  $v \in V - S$ .
- Thus,  $V - S$  covers  $(u, v)$ .

$\Leftarrow$

- Let  $V - S$  be any vertex cover.
- Consider two nodes  $u \in S$  and  $v \in S$ .
- Observe that  $(u, v) \notin E$  since  $V - S$  is a vertex cover.
- Thus, no two nodes in  $S$  are joined by an edge  $\Rightarrow S$  independent set. ■

# Reduction from Special Case to General Case

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Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

# Set Cover

**SET COVER:** Given a set  $U$  of elements, a collection  $S_1, S_2, \dots, S_m$  of subsets of  $U$ , and an integer  $k$ , does there exist a collection of  $\leq k$  of these sets whose union is equal to  $U$ ?

## Sample application.

- $m$  available pieces of software.
- Set  $U$  of  $n$  capabilities that we would like our system to have.
- The  $i$ th piece of software provides the set  $S_i \subseteq U$  of capabilities.
- Goal: achieve all  $n$  capabilities using fewest pieces of software.

Ex:

$$U = \{1, 2, 3, 4, 5, 6, 7\}$$

$$k = 2$$

$$S_1 = \{3, 7\}$$

$$S_4 = \{2, 4\}$$

$$S_2 = \{3, 4, 5, 6\}$$

$$S_5 = \{5\}$$

$$S_3 = \{1\}$$

$$S_6 = \{1, 2, 6, 7\}$$

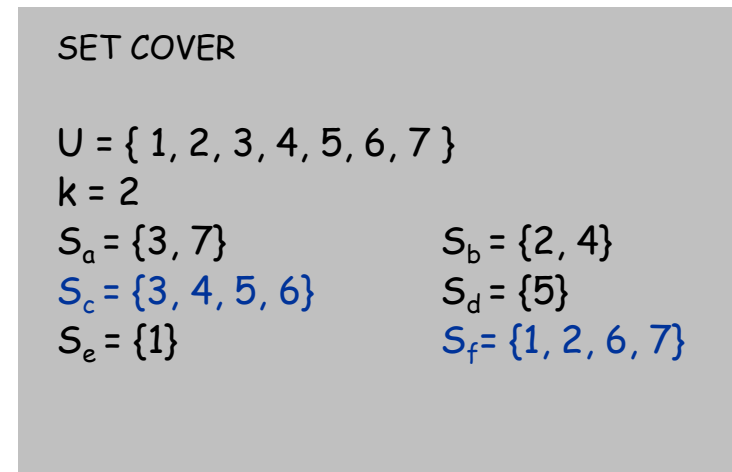
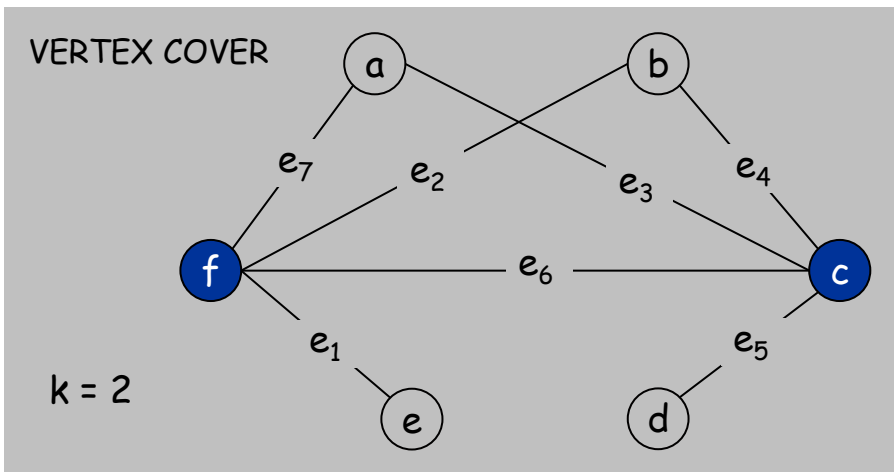
# Vertex Cover Reduces to Set Cover

**Claim.**  $\text{VERTEX-COVER} \leq_p \text{SET-COVER}$ .

**Pf.** Given a VERTEX-COVER instance  $G = (V, E)$ ,  $k$ , we construct a set cover instance whose size equals the size of the vertex cover instance.

## Construction.

- Create SET-COVER instance:
  - $k = k$ ,  $U = E$ ,  $S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size  $\leq k$  iff vertex cover of size  $\leq k$ . ■



## 8.2 Reduction via "Gadgets"

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Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."



# Satisfiability

**Literal:** A Boolean variable or its negation.

$$x_i \text{ or } \overline{x_i}$$

**Clause:** A disjunction of literals.

$$C_j = x_1 \vee \overline{x_2} \vee x_3$$

**Conjunctive normal form:** A propositional formula  $\Phi$  that is the conjunction of clauses.

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$$

**SAT:** Given CNF formula  $\Phi$ , does it have a satisfying truth assignment?

**3-SAT:** SAT where each clause contains exactly 3 literals.

↑  
each corresponds to a different variable

**Ex:** 
$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

**Yes:**  $x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false}, x_4 = \text{false}.$

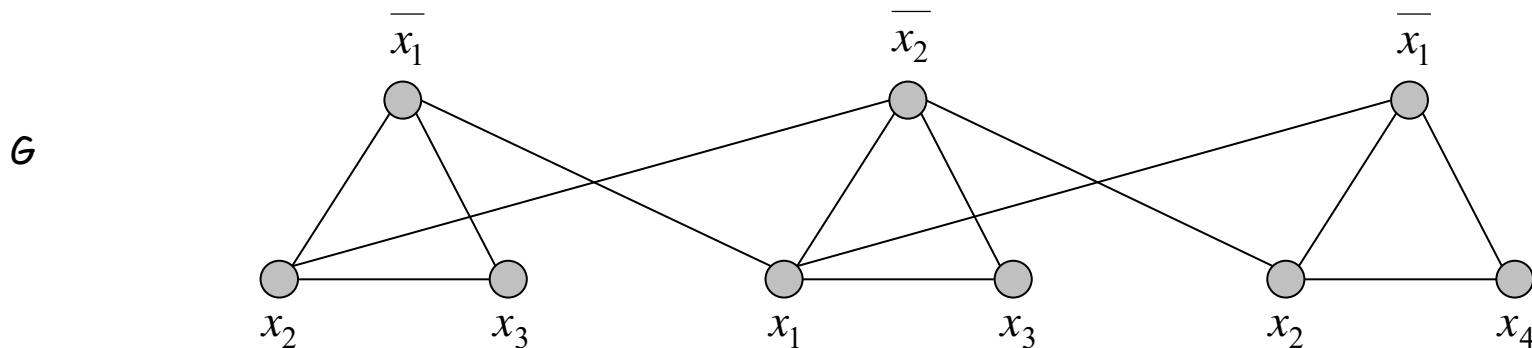
### 3 Satisfiability Reduces to Independent Set

**Claim.**  $3\text{-SAT} \leq_p \text{INDEPENDENT-SET}$ .

**Pf.** Given an instance  $\Phi$  of 3-SAT, we construct an instance  $(G, k)$  of INDEPENDENT-SET that has an independent set of size  $k$  iff  $\Phi$  is satisfiable.

**Construction.**

- $G$  contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.



$k = 3$

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

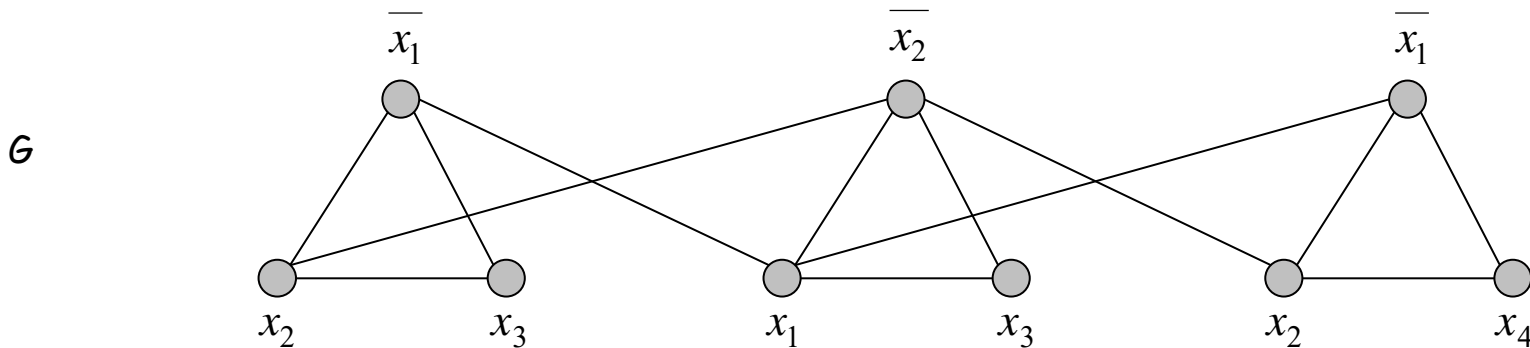
### 3 Satisfiability Reduces to Independent Set

**Claim.**  $G$  contains independent set of size  $k = |\Phi|$  iff  $\Phi$  is satisfiable.

**Pf.**  $\Rightarrow$  Let  $S$  be independent set of size  $k$ .

- $S$  must contain exactly one vertex in each triangle.
- Set these literals to true.  $\leftarrow$  and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

**Pf**  $\Leftarrow$  Given satisfying assignment, select one true literal from each triangle. This is an independent set of size  $k$ . ■



$k = 3$

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

# Review

## Basic reduction strategies.

- Simple equivalence:  $\text{INDEPENDENT-SET} \equiv_p \text{VERTEX-COVER}$ .
- Special case to general case:  $\text{VERTEX-COVER} \leq_p \text{SET-COVER}$ .
- Encoding with gadgets:  $3\text{-SAT} \leq_p \text{INDEPENDENT-SET}$ .

**Transitivity.** If  $X \leq_p Y$  and  $Y \leq_p Z$ , then  $X \leq_p Z$ .

**Pf idea.** Compose the two algorithms.

**Ex:**  $3\text{-SAT} \leq_p \text{INDEPENDENT-SET} \leq_p \text{VERTEX-COVER} \leq_p \text{SET-COVER}$ .

# Self-Reducibility

**Decision problem.** Does there **exist** a vertex cover of size  $\leq k$ ?

**Search problem.** **Find** vertex cover of minimum cardinality.

**Self-reducibility.** Search problem  $\leq_p$  decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

**Ex:** to find min cardinality vertex cover.

- (Binary) search for cardinality  $k^*$  of min vertex cover.
- Find a vertex  $v$  such that  $G - \{v\}$  has a vertex cover of size  $\leq k^* - 1$ .
  - any vertex in any min vertex cover will have this property
- Include  $v$  in the vertex cover.
- Recursively find a min vertex cover in  $G - \{v\}$ .

↑  
delete  $v$  and all incident edges