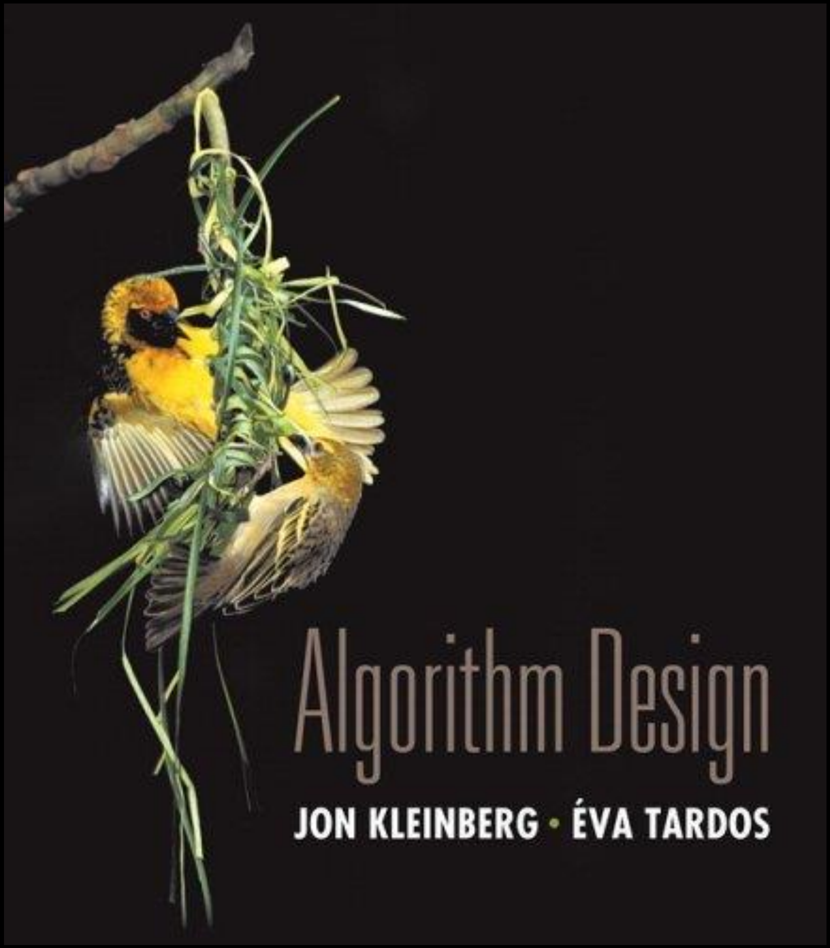


Chapter 10

Extending the Limits of Tractability

Slides by Kevin Wayne.

Coping With NP-Completeness

Q. Suppose I need to solve an NP-complete problem. What should I do?
A. Theory says you're unlikely to find poly-time algorithm.

Must sacrifice one of three desired features.

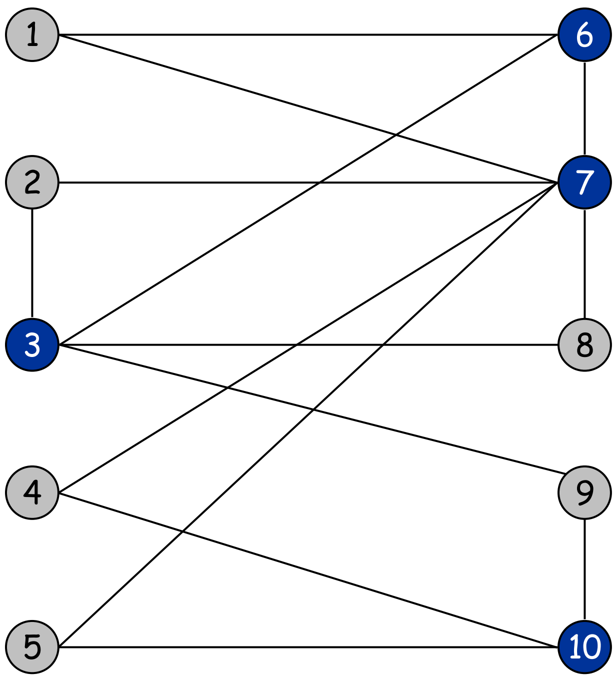
- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

This lecture. Solve some special cases of NP-complete problems that arise in practice.

10.1 Finding Small Vertex Covers

Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge (u, v) either $u \in S$, or $v \in S$, or both.



$k = 4$
$S = \{ 3, 6, 7, 10 \}$

Finding Small Vertex Covers

Q. What if k is small?

Brute force. $O(k\, n^{k+1})$.

- Try all $C(n, k) = O(n^k)$ subsets of size k.
- Takes $O(k\, n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on k, e.g., to $O(2^k\, k\, n)$.

Ex. n = 1,000, k = 10.
Brute. $k\, n^{k+1} = 10^{34} \Rightarrow$ infeasible.
Better. $2^k\, k\, n = 10^7 \Rightarrow$ feasible.

Remark. If k is a constant, algorithm is poly-time; if k is a small constant, then it's also practical.

Finding Small Vertex Covers

Claim. Let u-v be an edge of G. G has a vertex cover of size $\leq k$ iff at least one of $G - \{u\}$ and $G - \{v\}$ has a vertex cover of size $\leq k-1$.

↖ delete v and all incident edges

Pf. $\Rightarrow$

- Suppose G has a vertex cover S of size $\leq k$.
- S contains either u or v (or both). Assume it contains u.
- $S - \{u\}$ is a vertex cover of $G - \{u\}$.

Pf. $\Leftarrow$

- Suppose S is a vertex cover of $G - \{u\}$ of size $\leq k-1$.
- Then $S \cup \{u\}$ is a vertex cover of G. ▪

Claim. If G has a vertex cover of size k, it has $\leq k(n-1)$ edges.

Pf. Each vertex covers at most $n-1$ edges. ▪

Finding Small Vertex Covers: Algorithm

Claim. The following algorithm determines if G has a vertex cover of size $\leq k$ in $O(2^k kn)$ time.

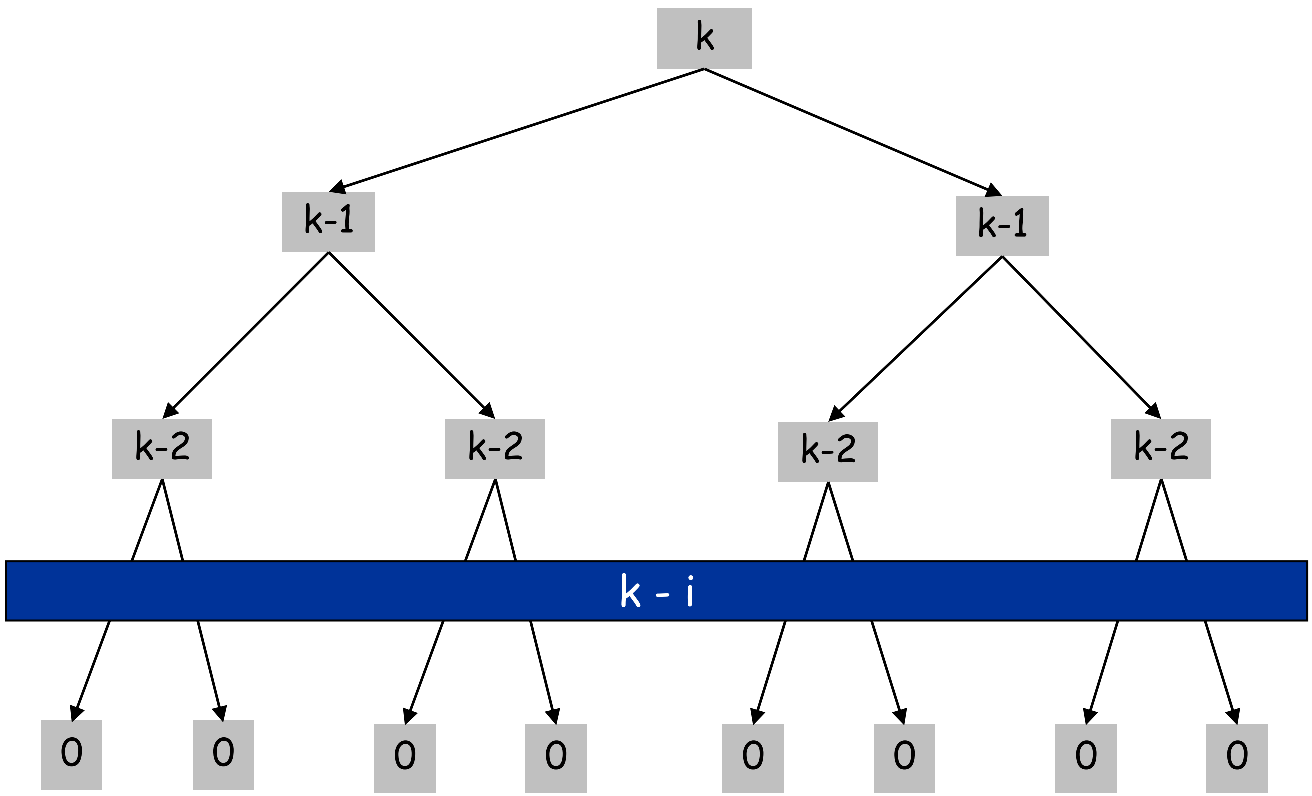
```
boolean Vertex-Cover(G, k) {
   if (G contains no edges)    return true
   if (G contains ≥ kn edges) return false

   let (u, v) be any edge of G
   a = Vertex-Cover(G - {u}, k-1)
   b = Vertex-Cover(G - {v}, k-1)
   return a or b
}
```

Pf.

- Correctness follows previous two claims.
- There are $\leq 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time. ▪

Finding Small Vertex Covers: Recursion Tree

$$T(n,k) \le \begin{cases} cn & \text{if } k = 1 \\ 2T(n,k-1) + ckn & \text{if } k > 1 \end{cases} \quad \Rightarrow \quad T(n,k) \le 2^k c\,k\,n$$



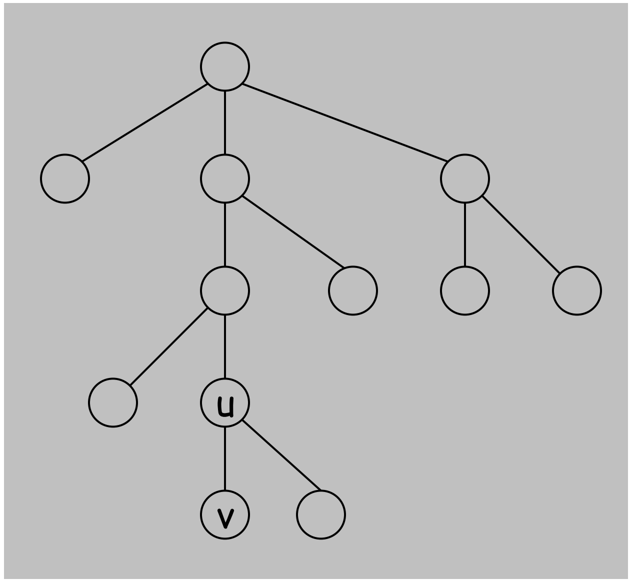
10.2 Solving NP-Hard Problems on Trees

Independent Set on Trees

Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two share an edge.

Fact. A tree on at least two nodes has at least two leaf nodes.

↖ degree = 1



Key observation. If v is a leaf, there exists a maximum size independent set containing v.

Pf. (exchange argument)

- Consider a max cardinality independent set S.
- If $v \in S$, we're done.
- If $u \notin S$ and $v \notin S$, then $S \cup \{v\}$ is independent $\Rightarrow$ S not maximum.
- IF $u \in S$ and $v \notin S$, then $S \cup \{v\} - \{u\}$ is independent. ▪

Independent Set on Trees: Greedy Algorithm

Theorem. The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

```
Independent-Set-In-A-Forest(F) {
   S ← φ
   while (F has at least one edge) {
      Let e = (u, v) be an edge such that v is a leaf
      Add v to S
      Delete from F nodes u and v, and all edges
         incident to them.
   }
   return S
}
```

Pf. Correctness follows from the previous key observation. ▪

Remark. Can implement in $O(n)$ time by considering nodes in postorder.

Weighted Independent Set on Trees

Weighted independent set on trees. Given a tree and node weights $w_v > 0$, find an independent set S that maximizes $\Sigma_{v \in S} w_v$.

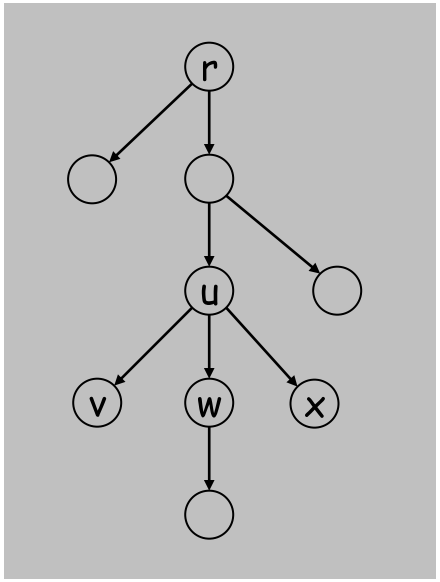
Observation. If (u, v) is an edge such that v is a leaf node, then either OPT includes u, or it includes all leaf nodes incident to u.

Dynamic programming solution. Root tree at some node, say r.

- $OPT_{in}(u)$ = max weight independent set rooted at u, containing u.
- $OPT_{out}(u)$ = max weight independent set rooted at u, not containing u.

$$OPT_{in}(u) = w_u + \sum_{v \in \text{children}(u)} OPT_{out}(v)$$

$$OPT_{out}(u) = \sum_{v \in \text{children}(u)} \max \{OPT_{in}(v),\ OPT_{out}(v)\}$$



children(u) = { v, w, x }

Independent Set on Trees: Greedy Algorithm

Theorem. The dynamic programming algorithm finds a maximum weighted independent set in trees in O(n) time.

```
Weighted-Independent-Set-In-A-Tree(T) {
   Root the tree at a node r
   foreach (node u of T in postorder) {
      if (u is a leaf) {
         M_in [u] = w_u
         M_out[u] = 0
      }
      else {
         M_in [u] = Σ_{v∈children(u)} M_out[v]  +  w_v
         M_out[u] = Σ_{v∈children(u)} max(M_out[v], M_in[v])
      }
   }
   return max(M_in[r], M_out[r])
}
```

↑ ensures a node is visited after all its children

Pf. Takes O(n) time since we visit nodes in postorder and examine each edge exactly once. ▪

10.3 Circular Arc Coloring

Wavelength-Division Multiplexing

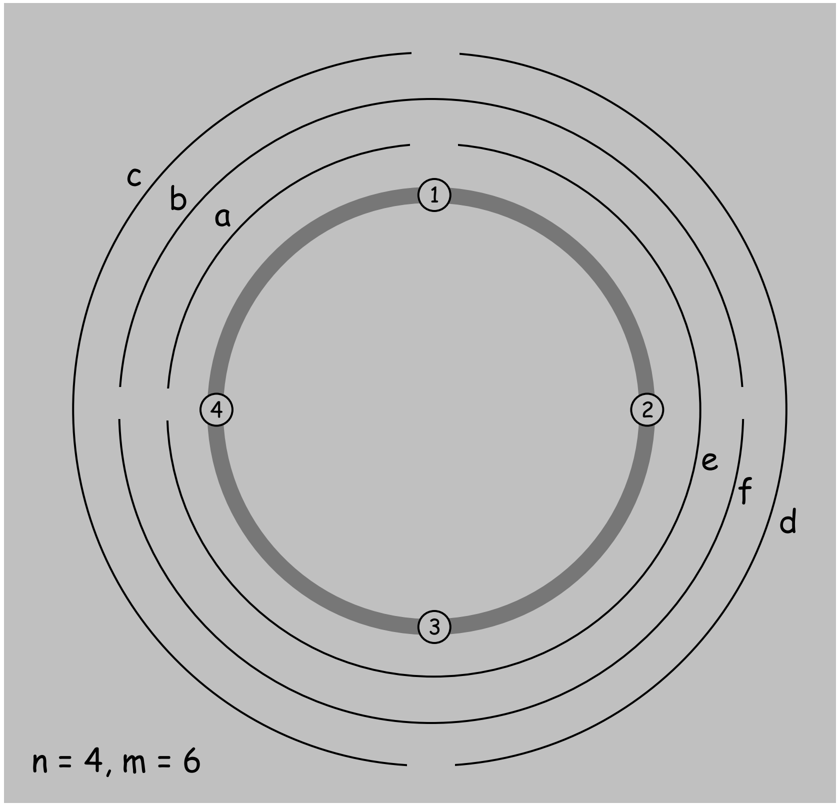
Wavelength-division multiplexing (WDM). Allows m communication streams (arcs) to share a portion of a fiber optic cable, provided they are transmitted using different wavelengths.

Ring topology. Special case is when network is a cycle on n nodes.

Bad news. NP-complete, even on rings.

Brute force. Can determine if k colors suffice in $O(k^m)$ time by trying all k-colorings.

Goal. $O(f(k)) \cdot \text{poly}(m, n)$ on rings.



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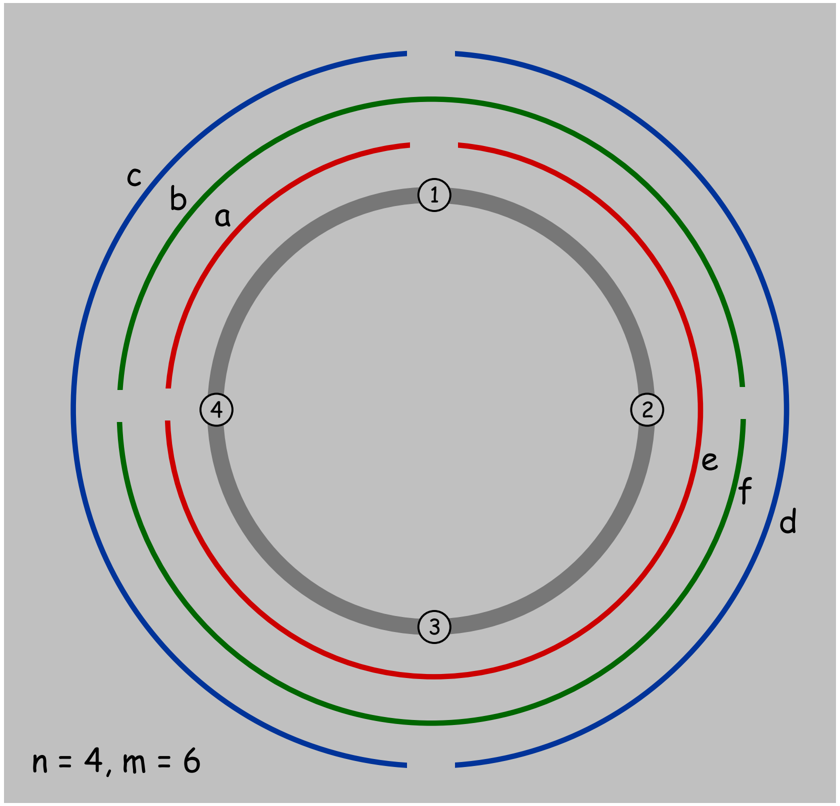
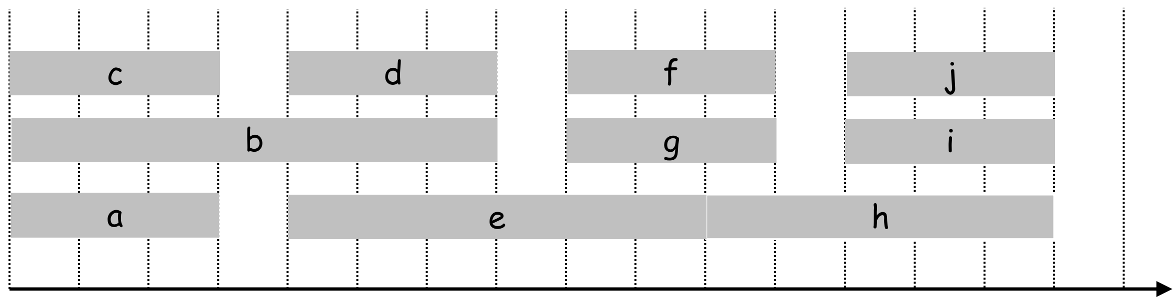


Review: Interval Coloring

Interval coloring. Greedy algorithm finds coloring such that number of colors equals depth of schedule.

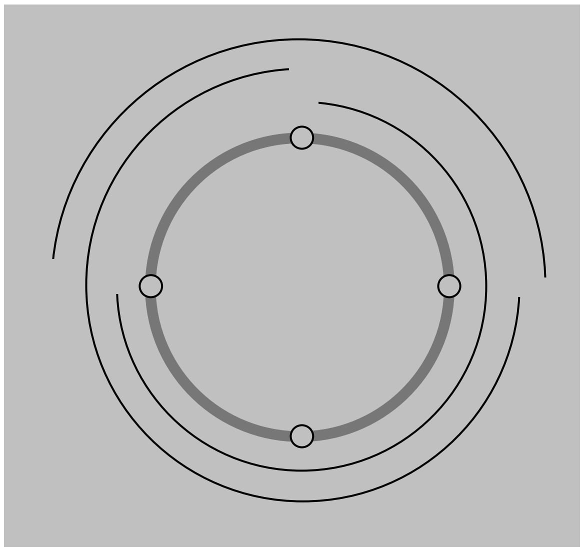
↖ maximum number of streams at one location



Circular arc coloring.

- Weak duality: number of colors $\geq$ depth.
- Strong duality does not hold.

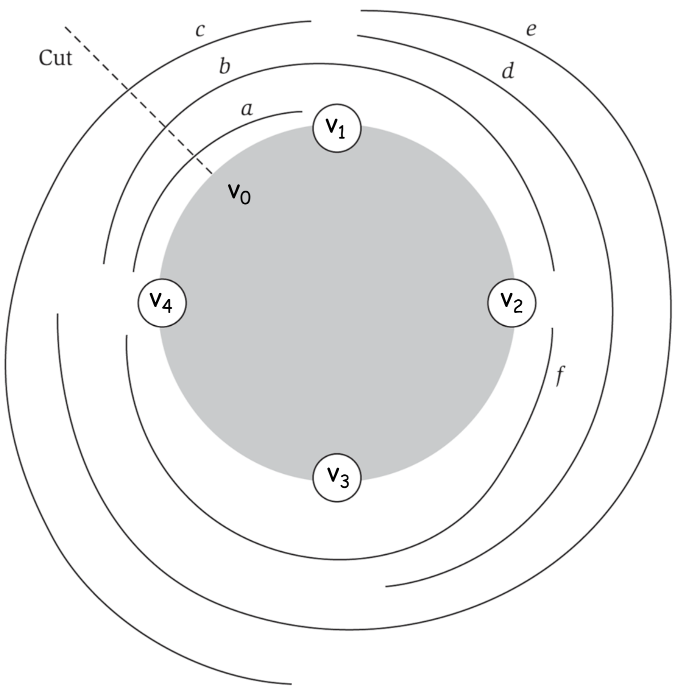
max depth = 2
min colors = 3



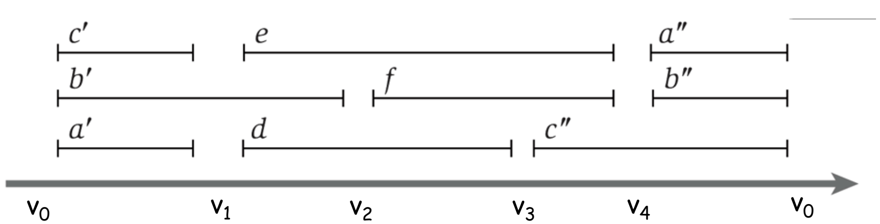
(Almost) Transforming Circular Arc Coloring to Interval Coloring

Circular arc coloring. Given a set of n arcs with depth $d \le k$, can the arcs be colored with k colors?

Equivalent problem. Cut the network between nodes v_1 and v_n. The arcs can be colored with k colors iff the intervals can be colored with k colors in such a way that "sliced" arcs have the same color.



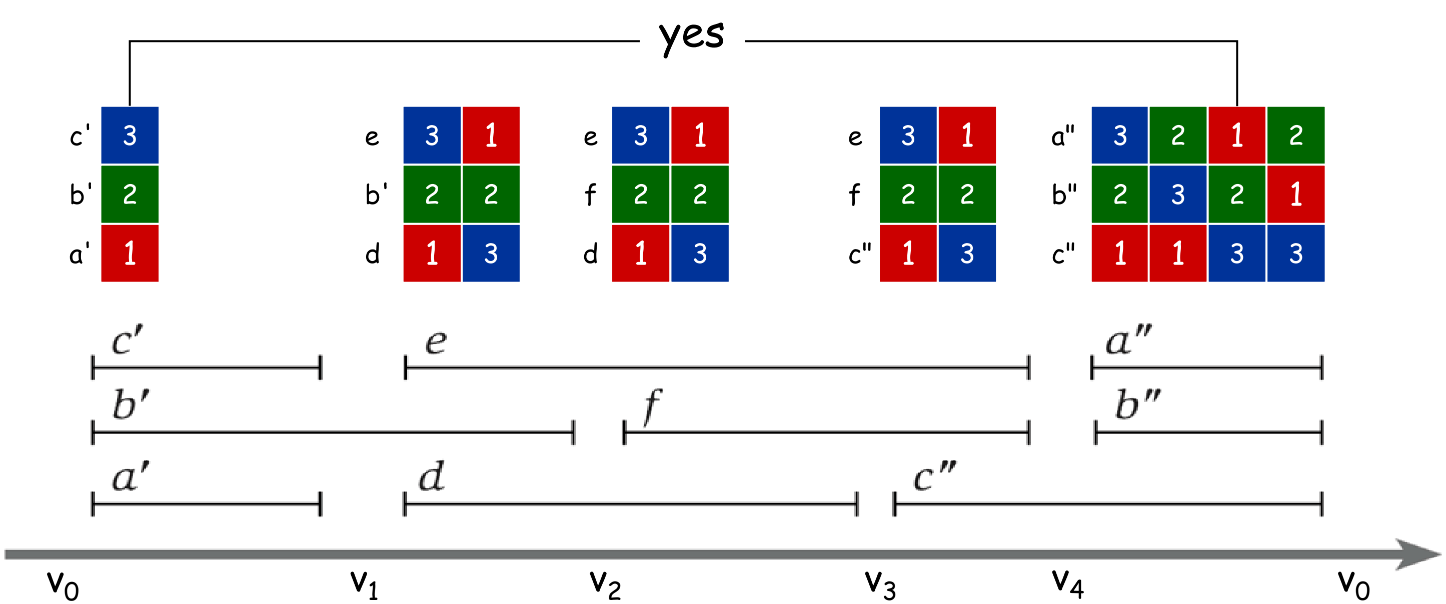
colors of a', b', and c' must correspond to colors of a", b", and c"



Circular Arc Coloring: Dynamic Programming Algorithm

Dynamic programming algorithm.

- Assign distinct color to each interval which begins at cut node v_0.
- At each node v_i, some intervals may finish, and others may begin.
- Enumerate all k-colorings of the intervals through v_i that are consistent with the colorings of the intervals through v_{i-1}.
- The arcs are k-colorable iff some coloring of intervals ending at cut node v_0 is consistent with original coloring of the same intervals.



Circular Arc Coloring: Running Time

Running time. $O(k! \cdot n)$.

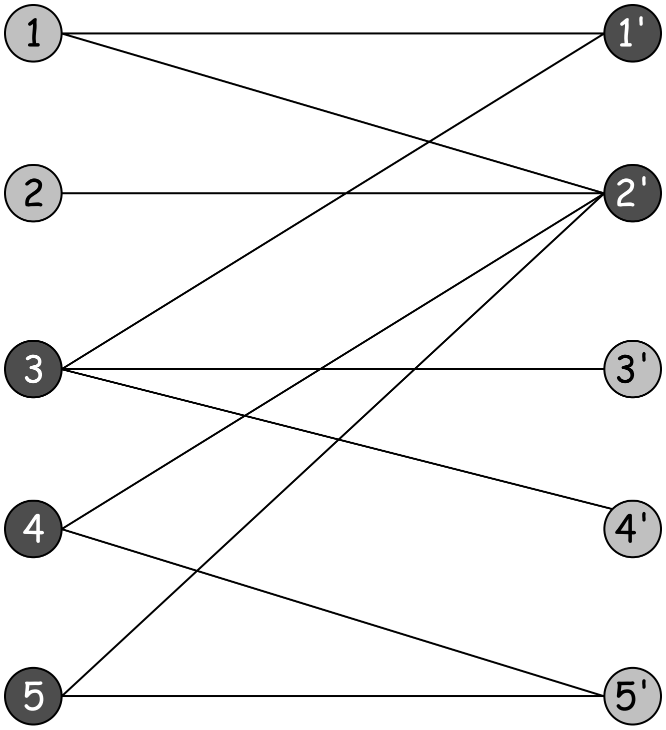
- n phases of the algorithm.
- Bottleneck in each phase is enumerating all consistent colorings.
- There are at most k intervals through v_i, so there are at most k! colorings to consider.

Remark. This algorithm is practical for small values of k (say k = 10) even if the number of nodes n (or paths) is large.

Vertex Cover in Bipartite Graphs

Vertex Cover

Vertex cover. Given an undirected graph $G = (V, E)$, a vertex cover is a subset of vertices $S \subseteq V$ such that for each edge $(u, v) \in E$, either $u \in S$ or $v \in S$ or both.

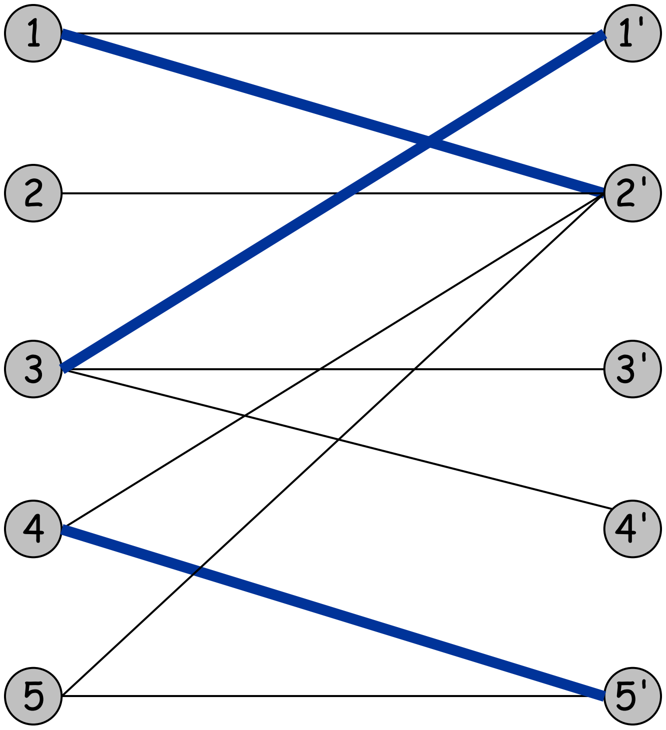


S = { 3, 4, 5, 1', 2' }
|S| = 5

Vertex Cover

Weak duality. Let M be a matching, and let S be a vertex cover.
Then, $|M| \leq |S|$.

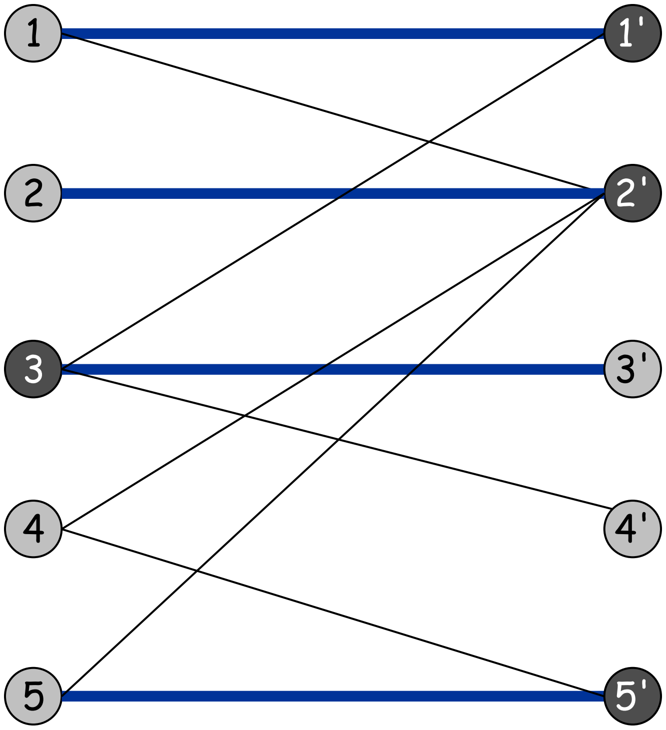
Pf. Each vertex can cover at most one edge in any matching.



M = 1-2', 3-1', 4-5'
|M| = 3

Vertex Cover: König-Egerváry Theorem

König-Egerváry Theorem. In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.



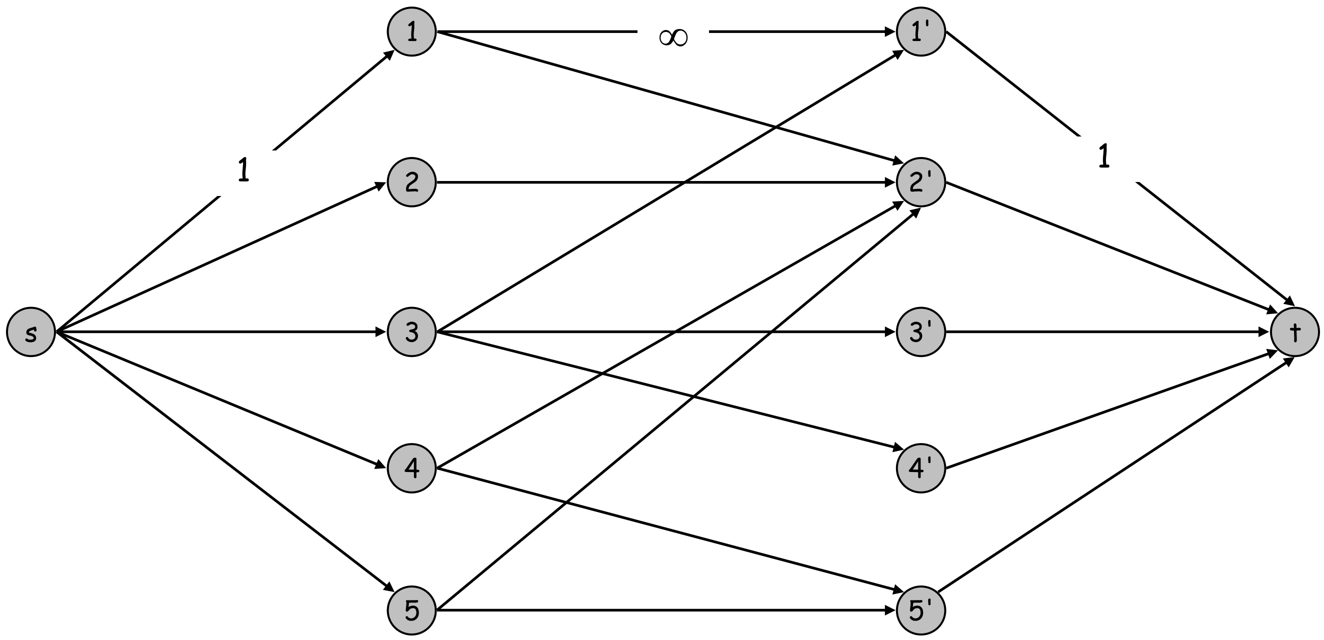
S* = { 3, 1', 2', 5'}
|S*| = 4

M* = 1-1', 2-2', 3-3', 5-5'
|M*| = 4

Vertex Cover: Proof of König-Egerváry Theorem

König-Egerváry Theorem. In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching M and cover S such that |M| = |S|.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.



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- Suffices to find matching M and cover S such that $|M| = |S|$.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$, $R_B = R \cap B$.

- Claim 1. $S = L_B \cup R_A$ is a vertex cover.
  - consider $(u, v) \in E$
  - $u \in L_A$, $v \in R_B$ impossible since infinite capacity
  - thus, either $u \in L_B$ or $v \in R_A$ or both

- Claim 2. $|S| = |M|$.
  - max-flow min-cut theorem $\Rightarrow$ $|M| = cap(A, B)$
  - only edges of form (s, u) or (v, t) contribute to cap(A, B)
  - $|M| = cap(A, B) = |L_B| + |R_A| = |S|$. ▪