TDT4121

Assignment Lecture Week 43

Network Flow

Main idea

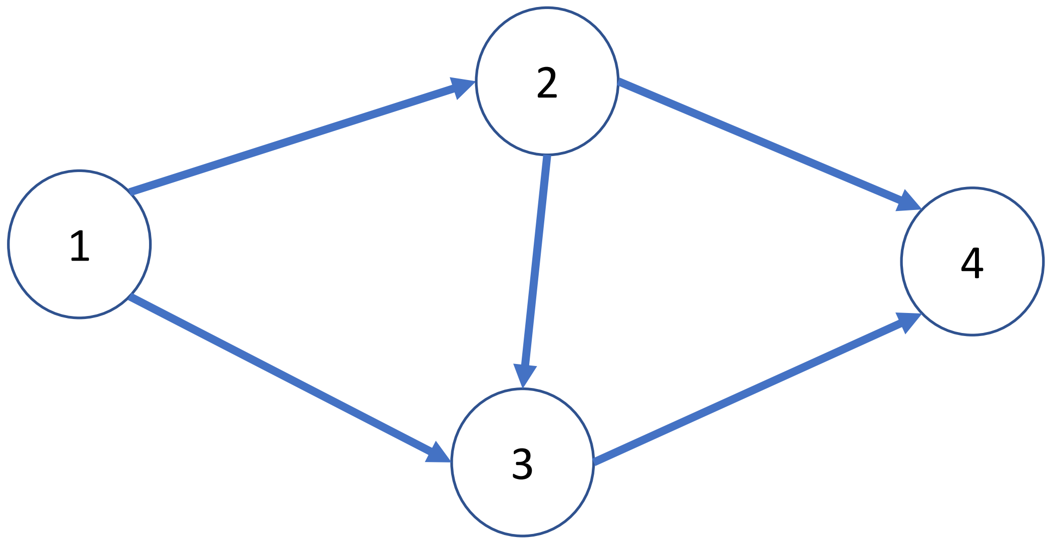
Lots of problems can be modeled as flow through a network

- Traffic flow, water flow, gas flow etc.

Network consists of nodes and edges

- Each edge has a capacity constraint
- There is a starting node (source) and a goal node (sink)

Network consists of nodes and directed edges



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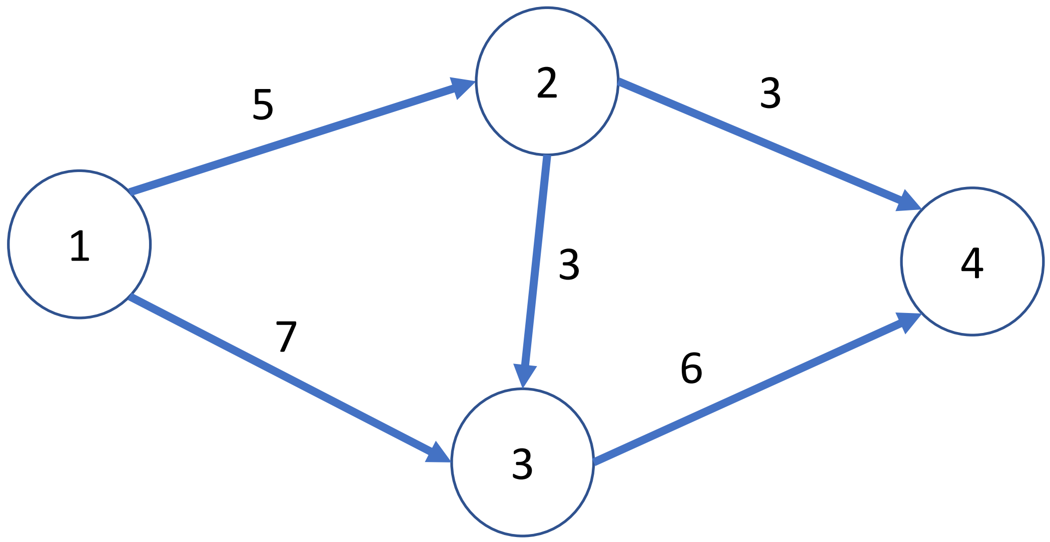

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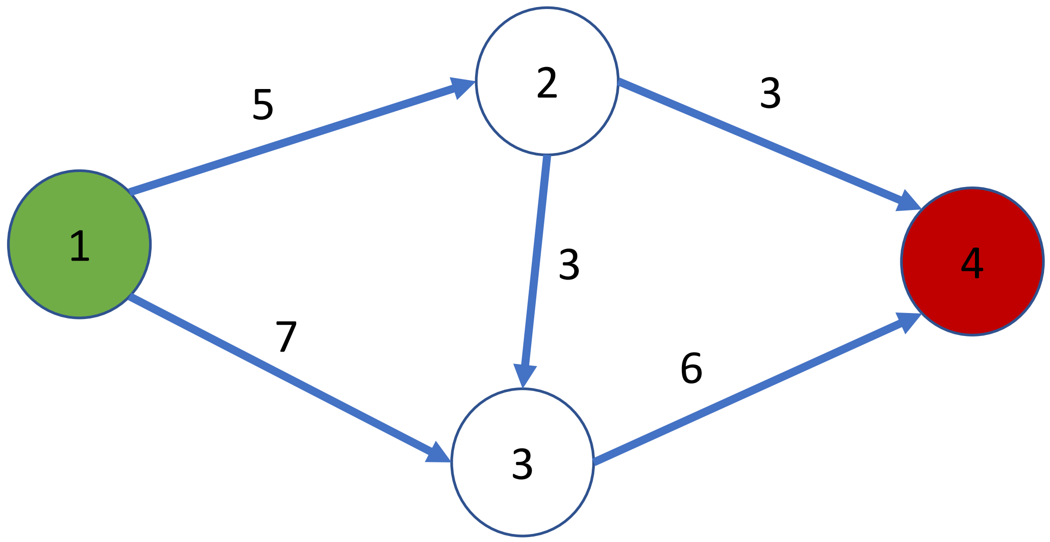
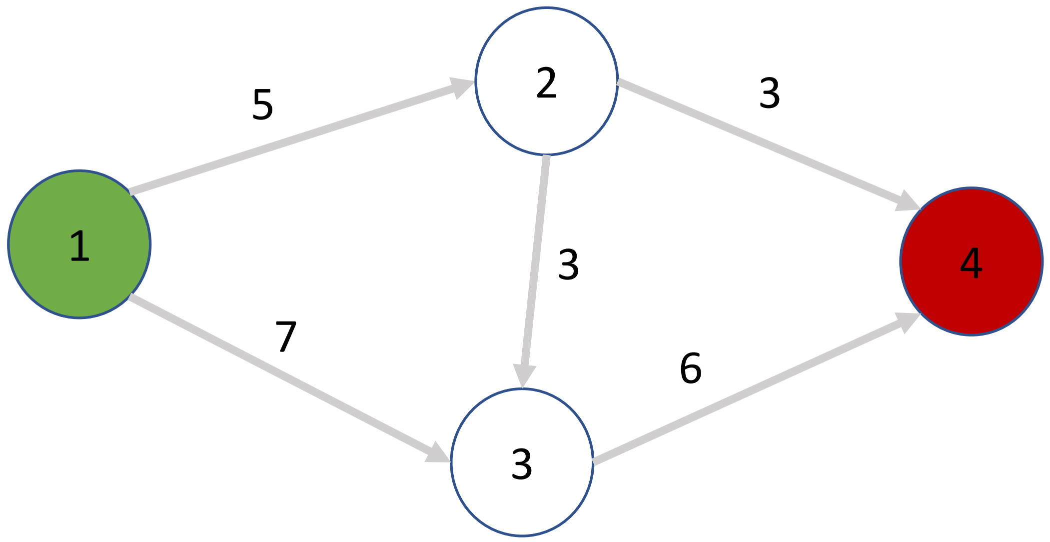


Problem: what is the maximum flow possible?

- Idea: send flow from 1 until maximum capacity is reached.



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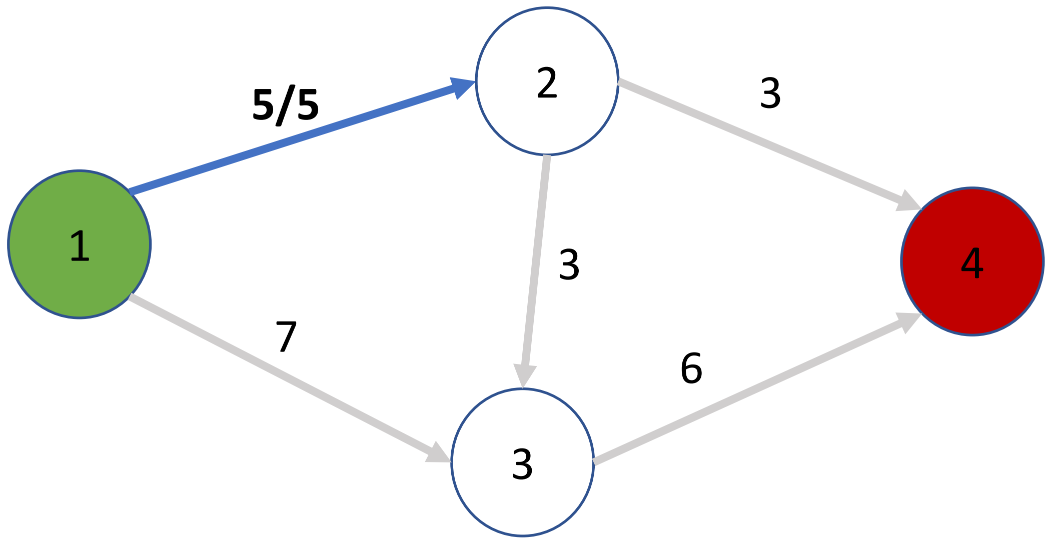

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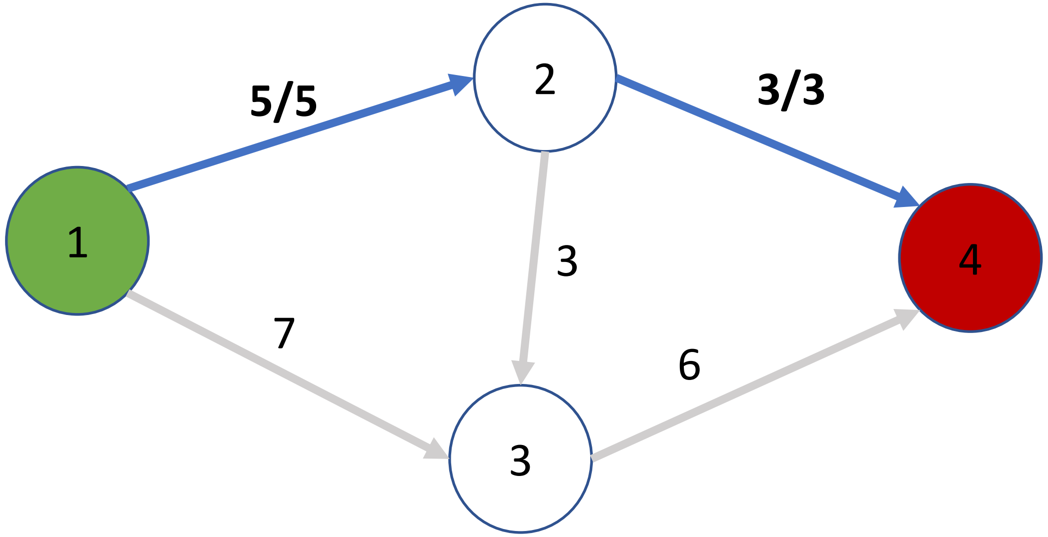

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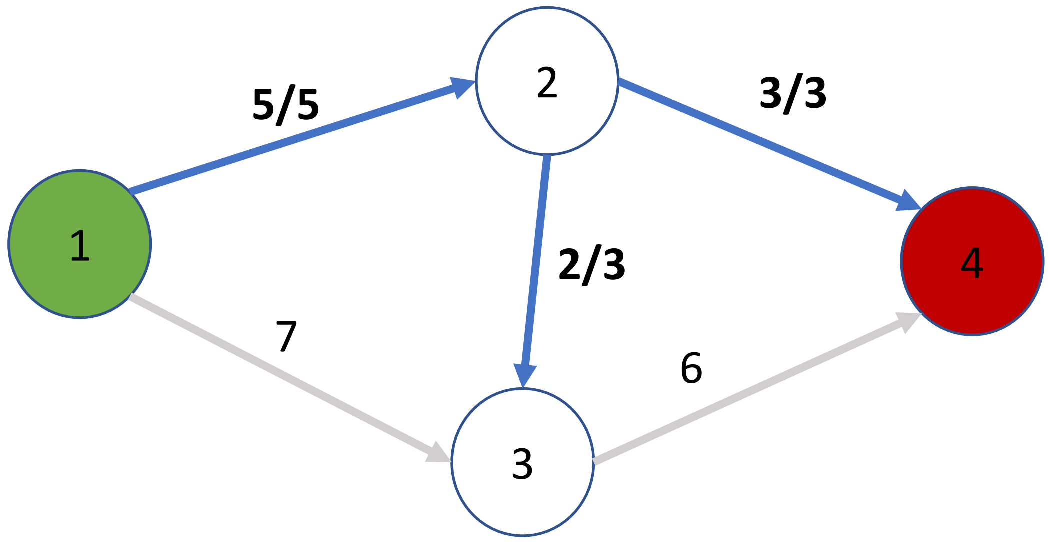

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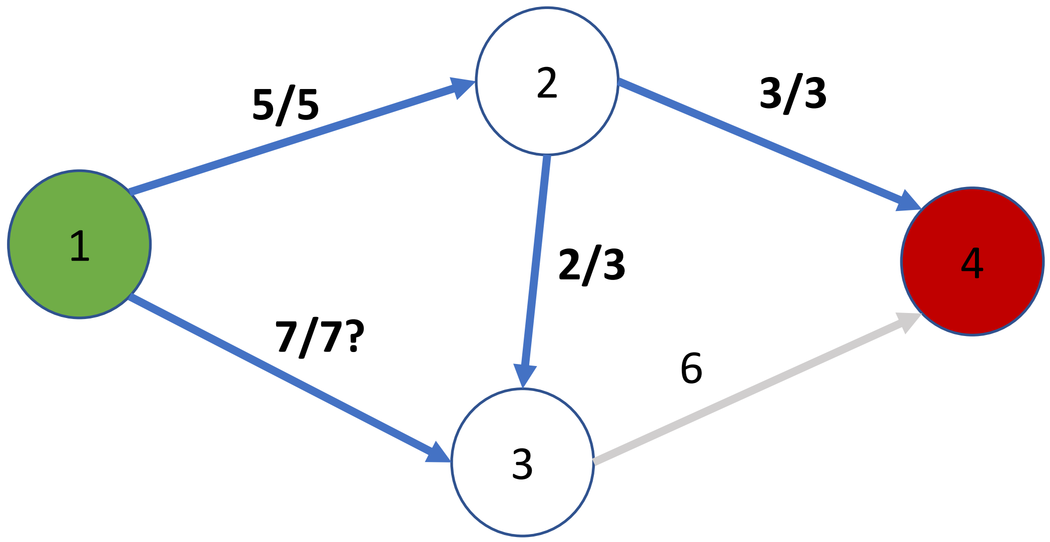

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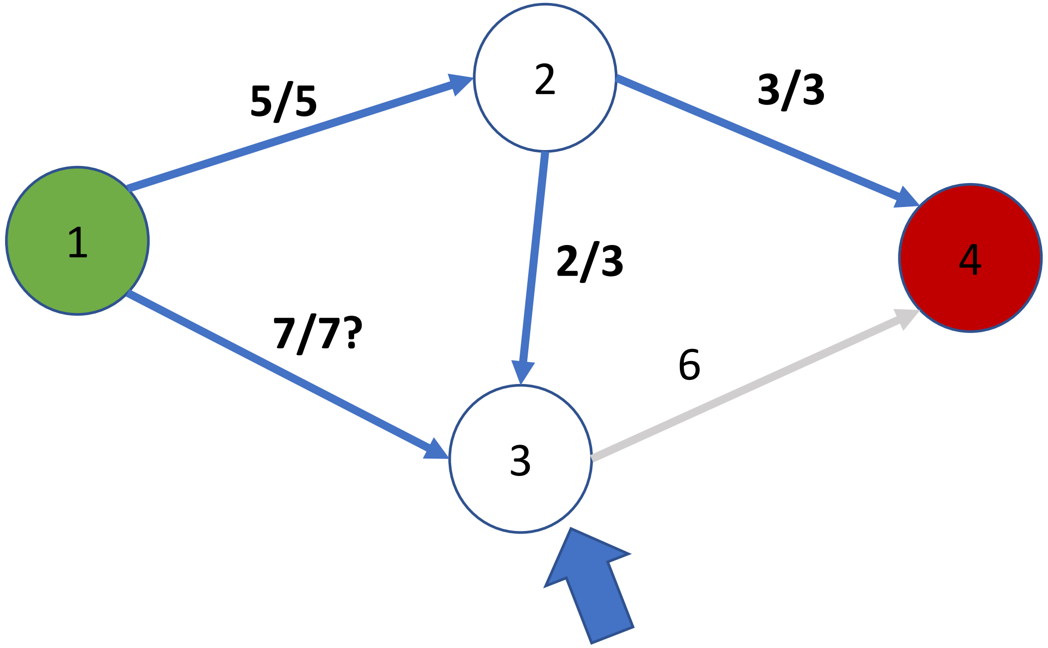
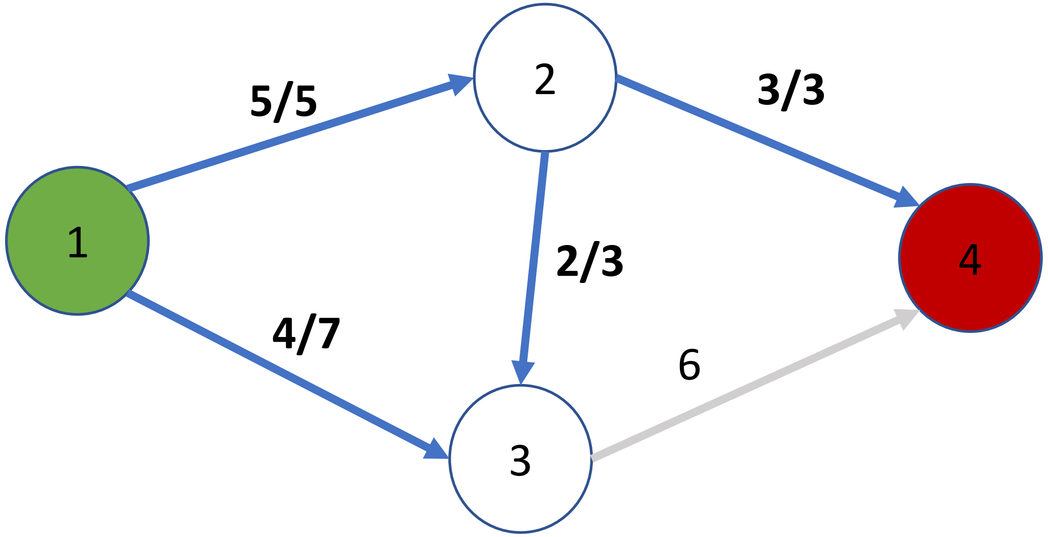


7+2=9 > 6, nowhere for this flow to go!

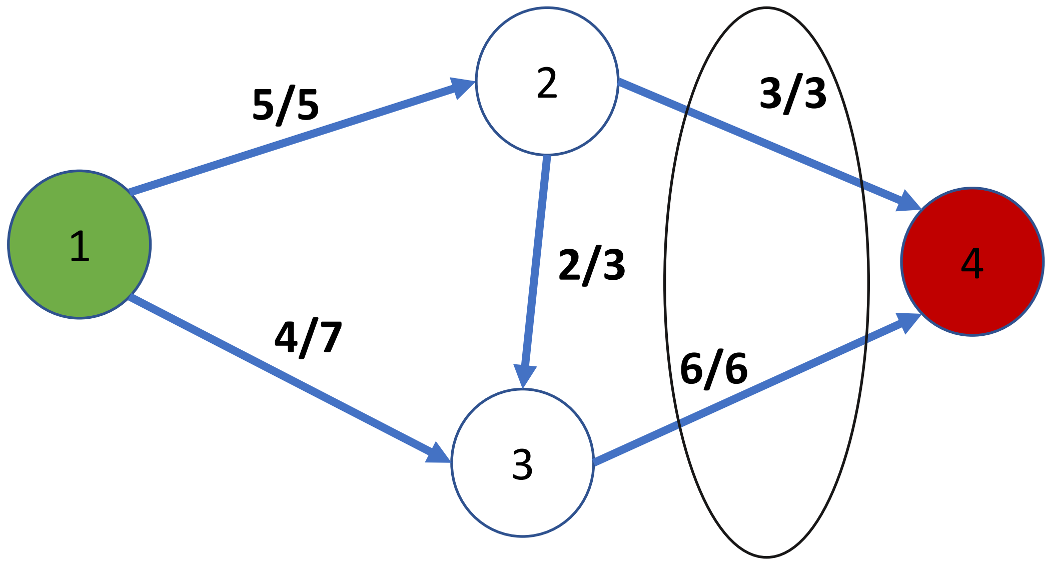
Problem: what is the maximum flow possible?

- Idea: send flow from 1 until maximum capacity is reached.



Max flow reached!

- The maximum flow equals the sum of all flow going into the sink (red)
- Max flow = 3 + 6 = 9



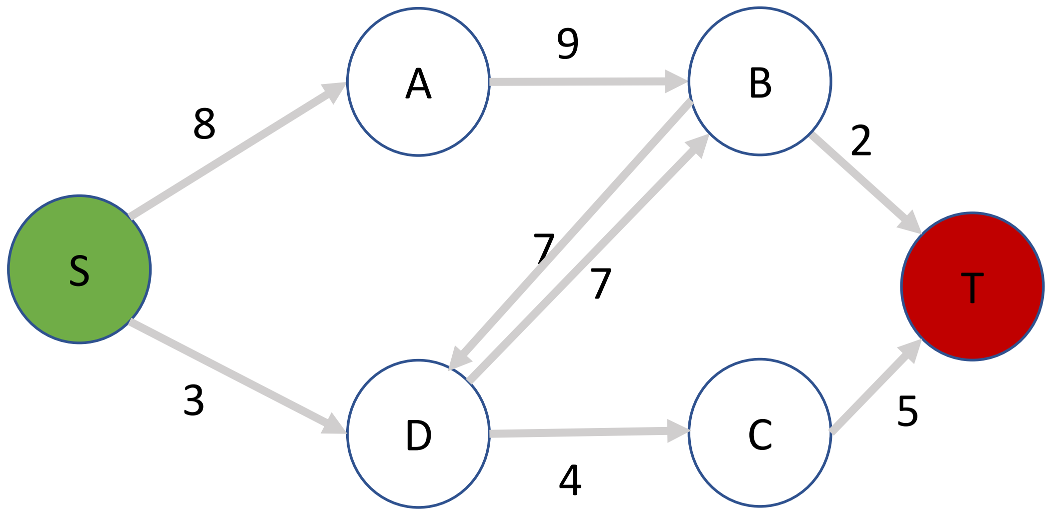
Trial-and-error takes too long for more complex graphs

- How can we find the maximum flow for any given network quickly?

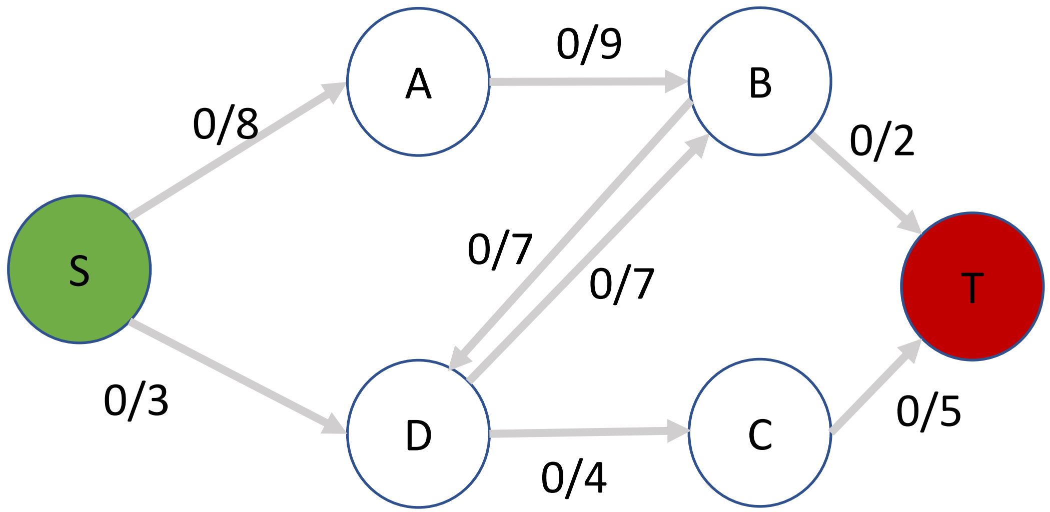
Solution: Ford Fulkerson Algorithm

- Main idea: as long as there is a path from start to finish (source to sink) with available flow, send flow along that path. Repeat this until there is no such path left

Example

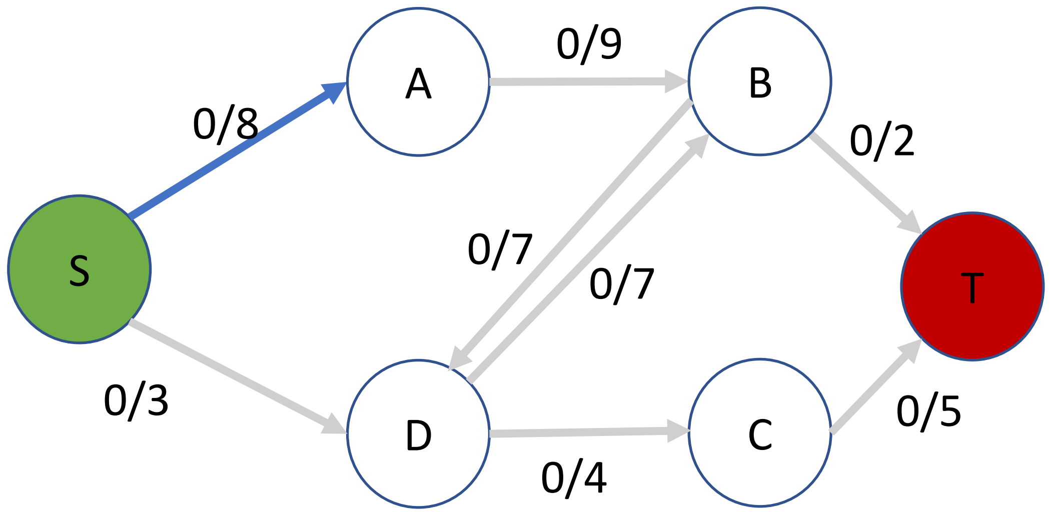


Step 1: set all flows to 0



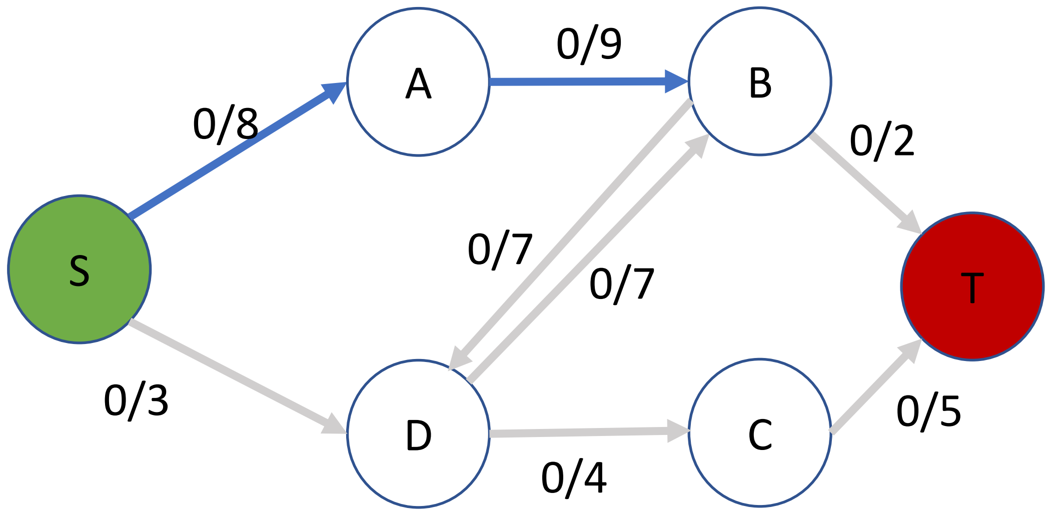
Step 2: find apath from S to T

S -> A



Step 2: find a path from S to T

S -> A -> B



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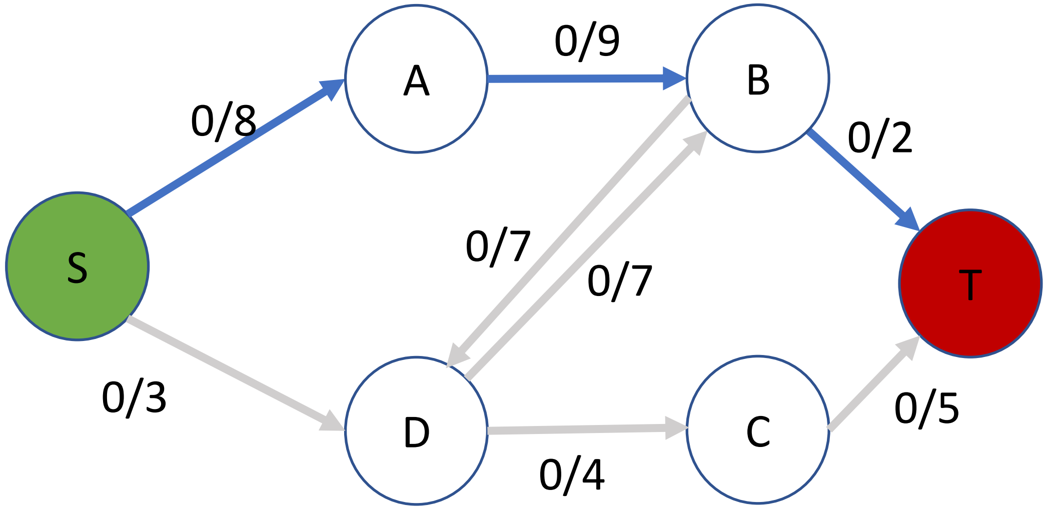
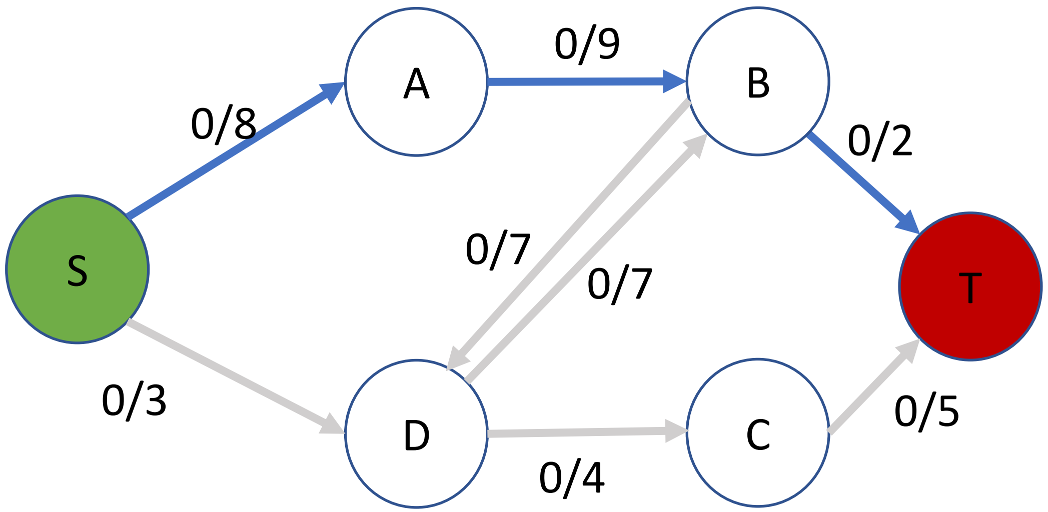
S -> A -> B -> T



Step 3: identify the minimum capacity along this path

S -> A -> B -> T



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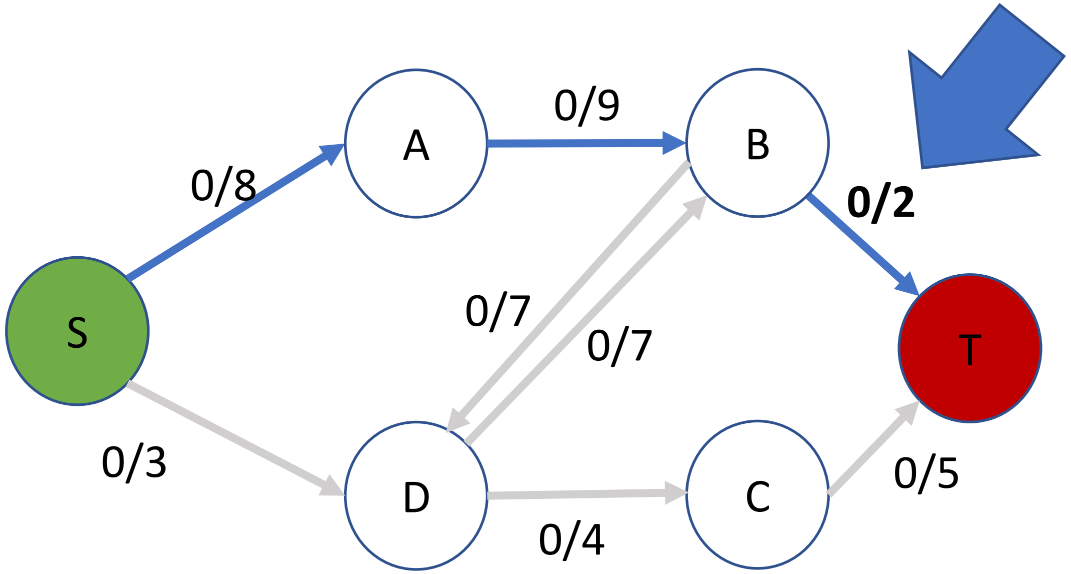
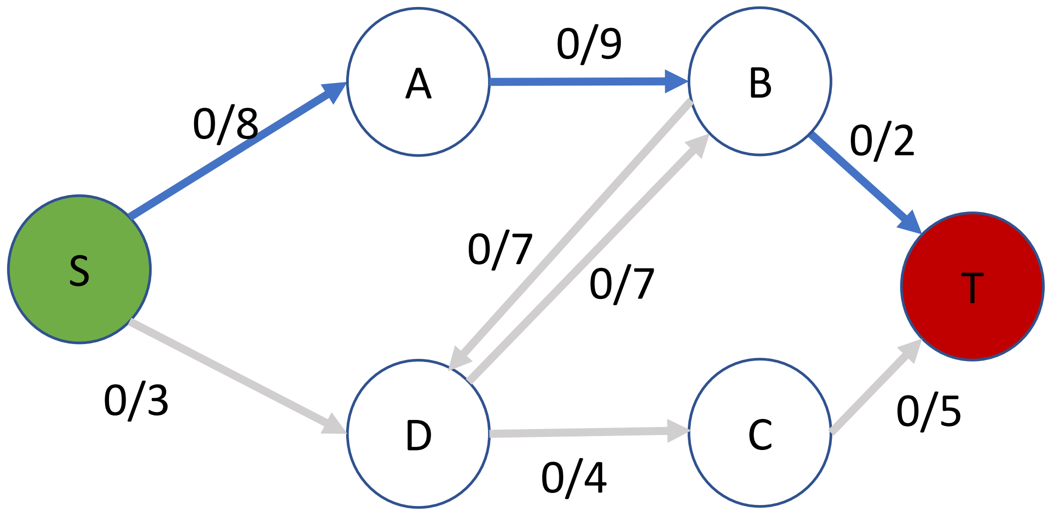


Step 4: send flow equal to the minimum capacity along this path

S -> A -> B -> T



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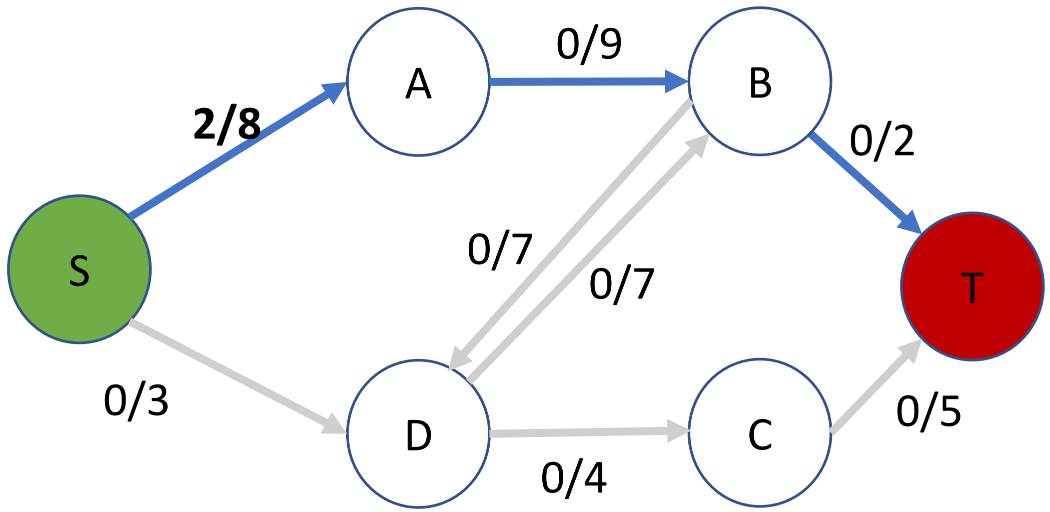

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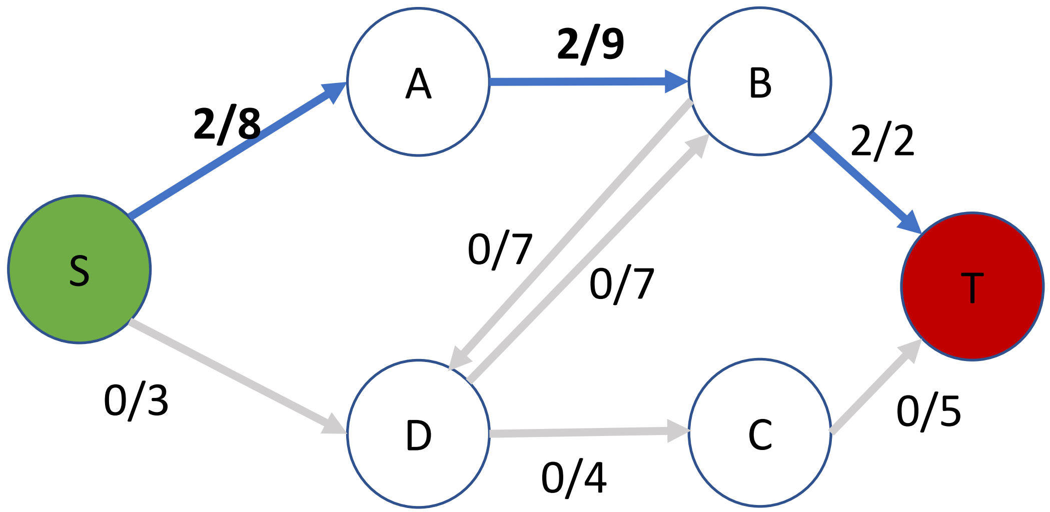

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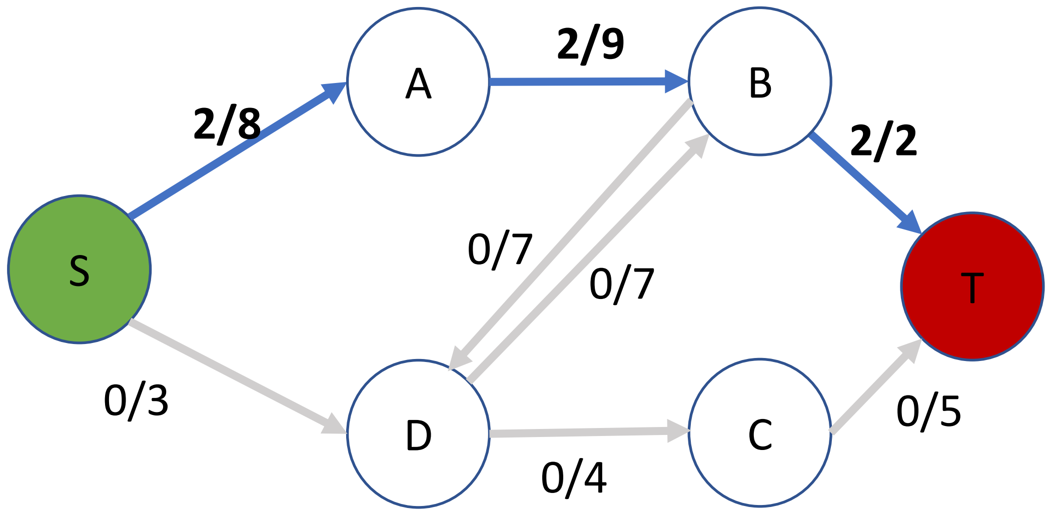
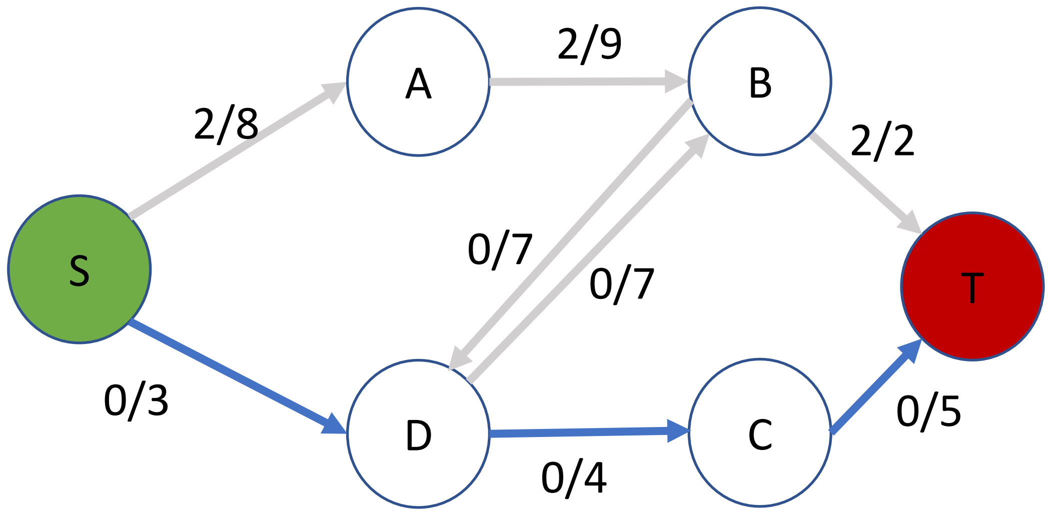


Repeat until there is no path left

1. Select path

S -> D -> C -> T



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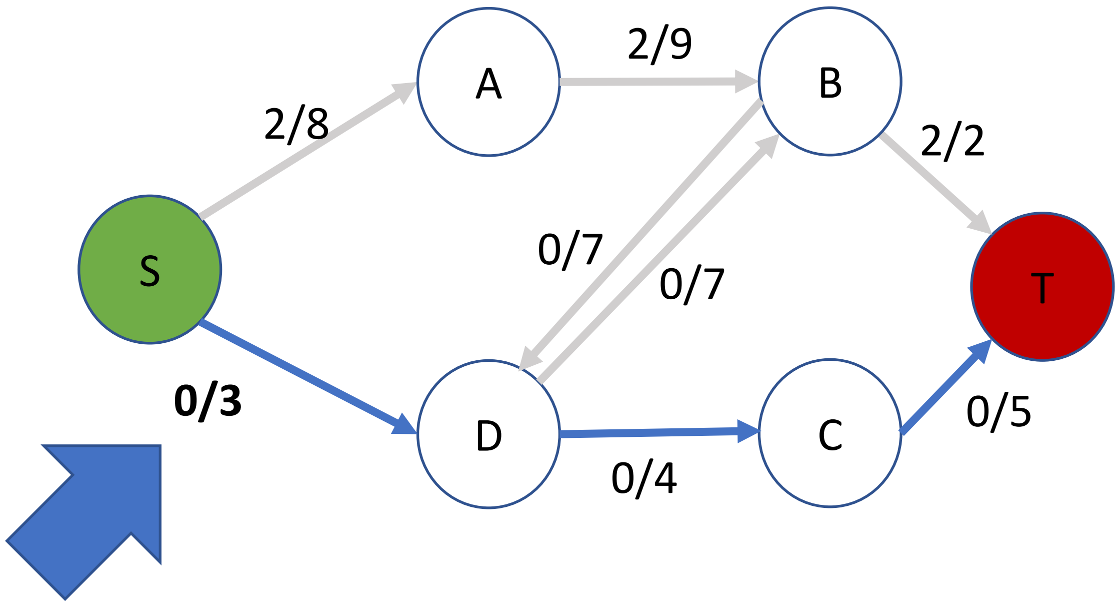
1. Select path
2. Find minimum capacity along path

S -> D -> C -> T



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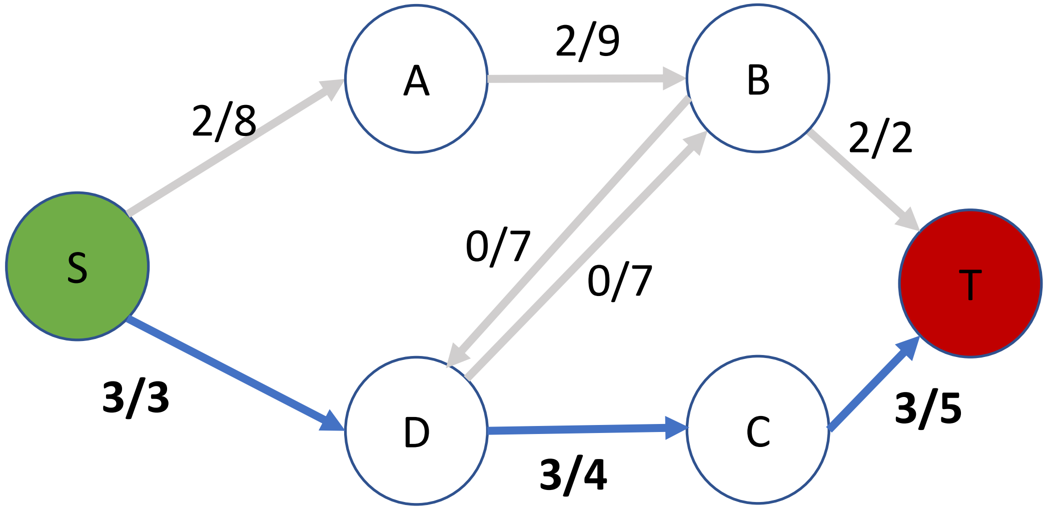

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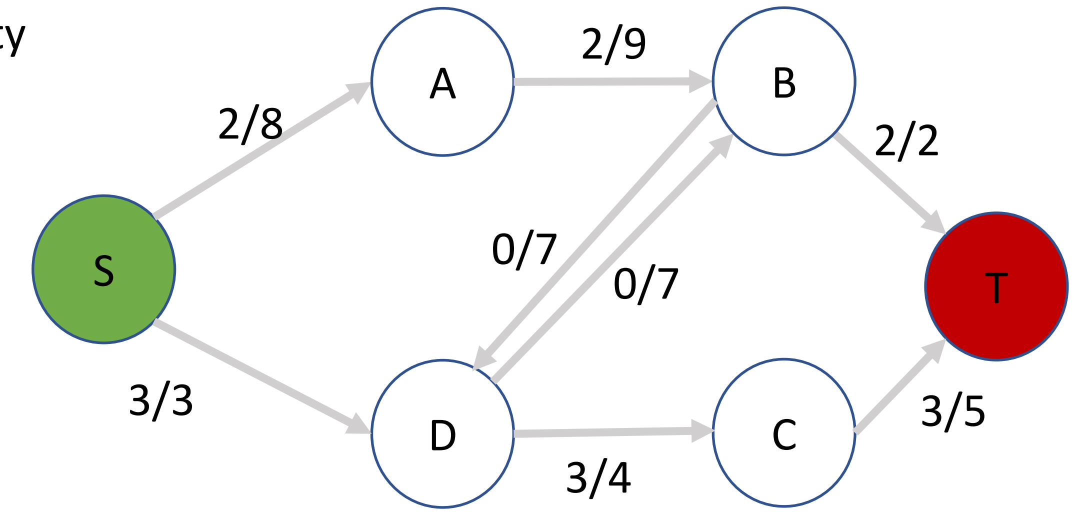

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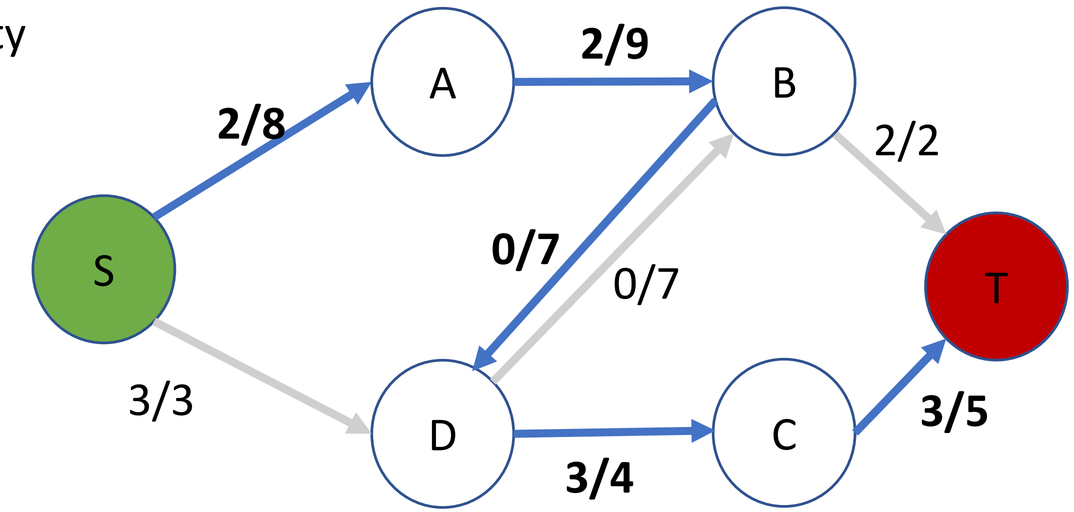

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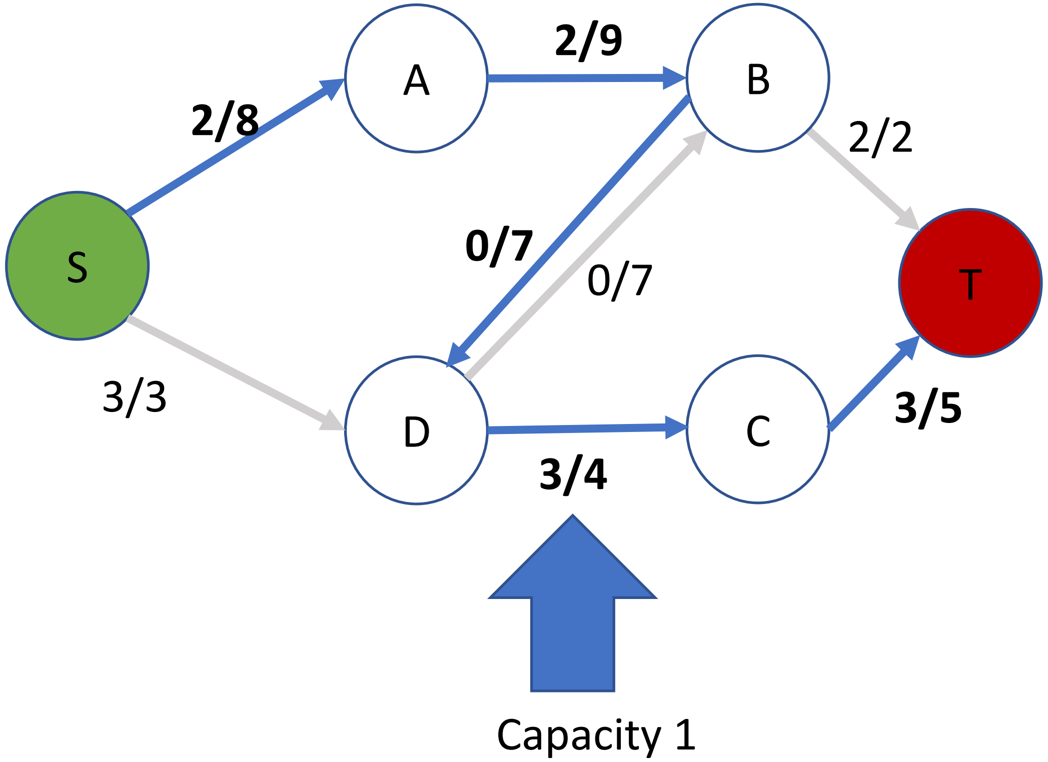

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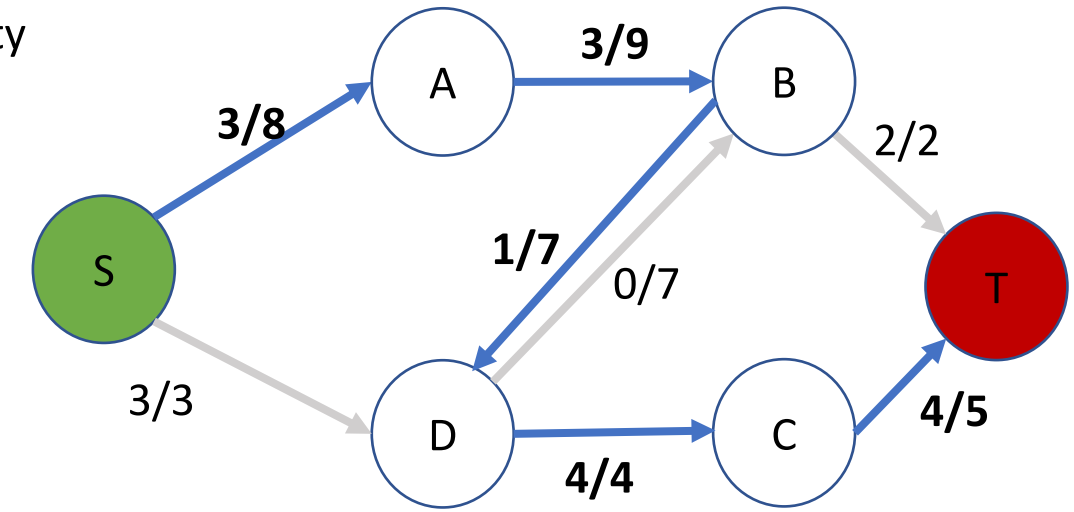
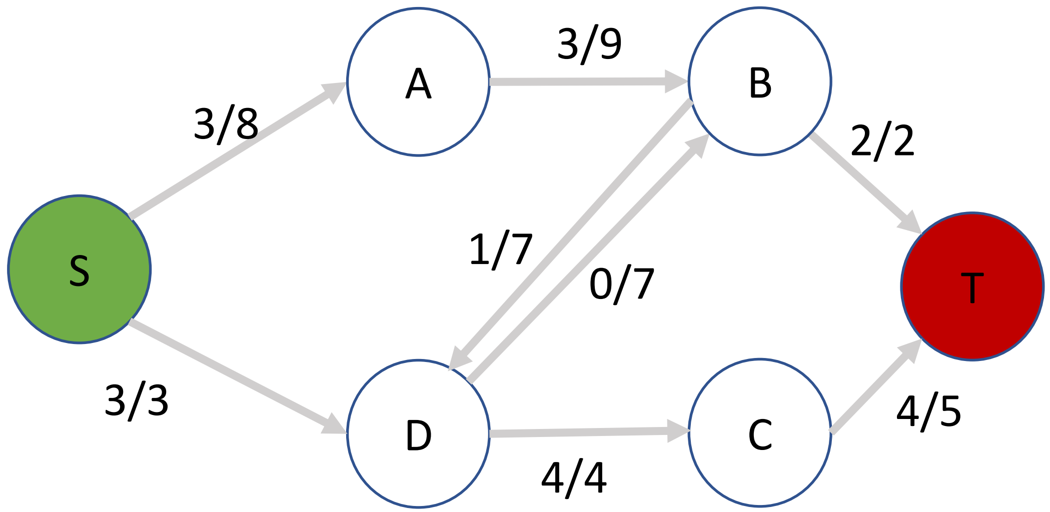


No more paths -> maximum flow reached



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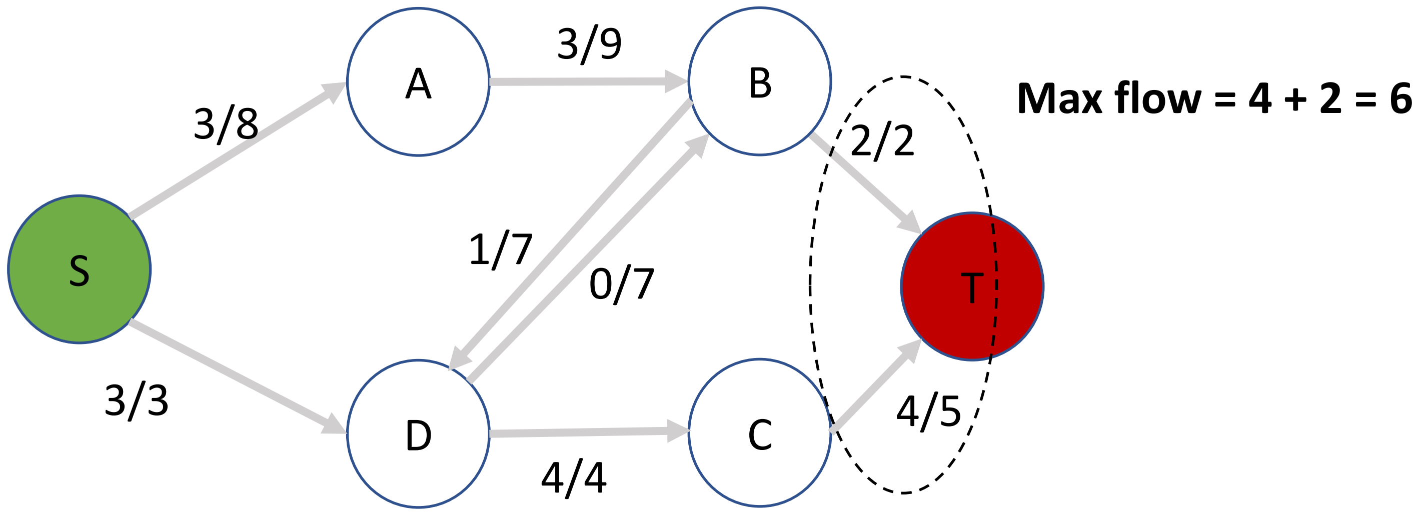
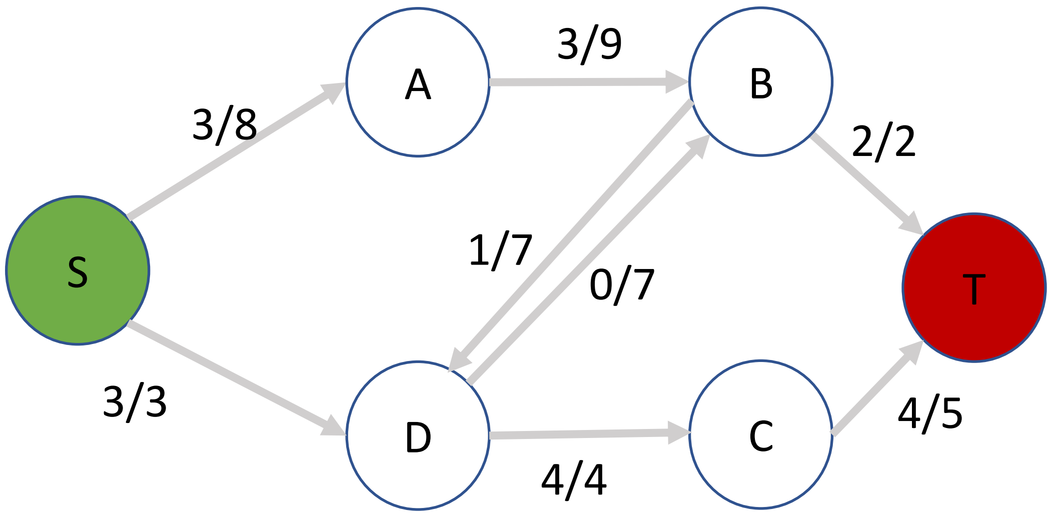


What is the time complexity of this?

1. Find augmenting path	*O(E) for DFS*
2. Identify the minimum capacity along the path	*O(1)*
3. Update the flow along the path	*O(1)*



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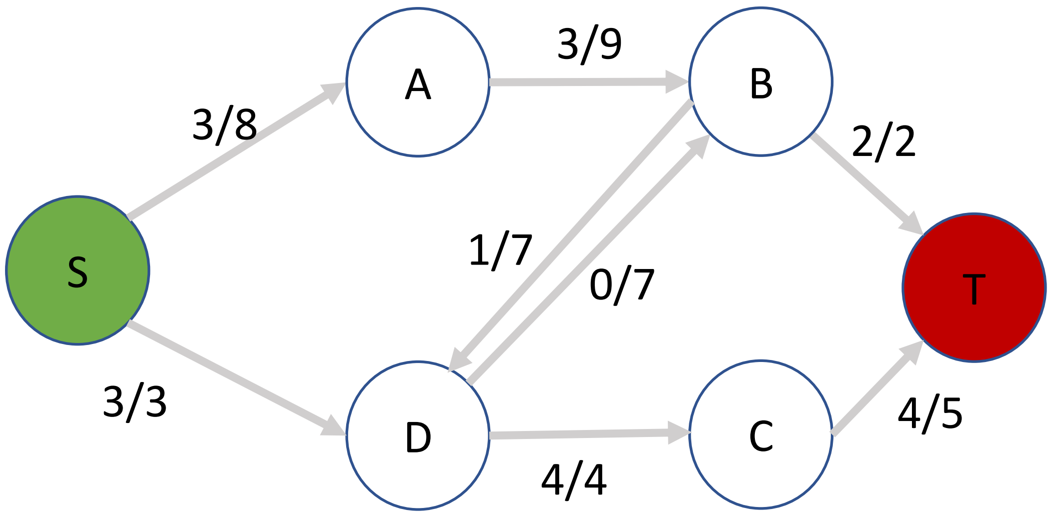
This is repeated at most f times, where f = max flow



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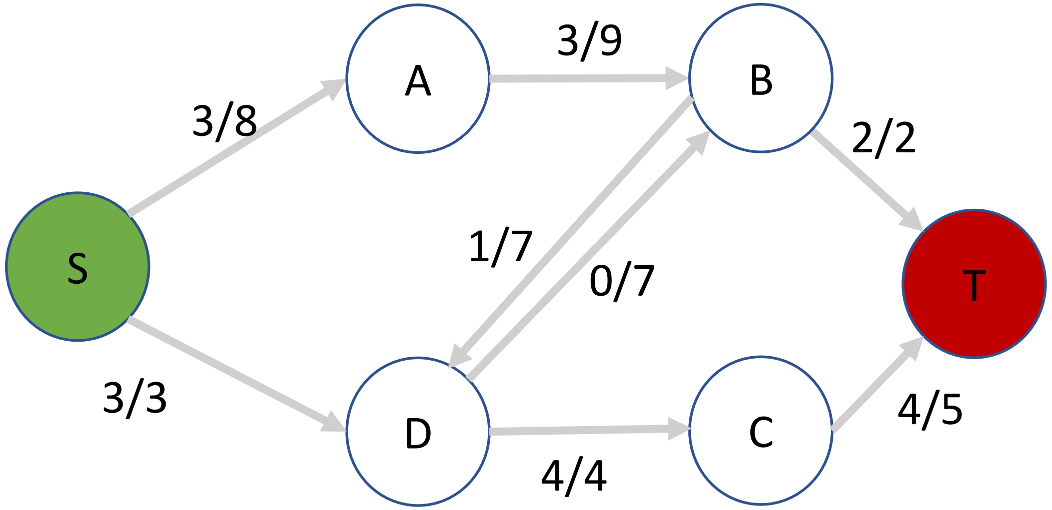
O((E+1+1)*f)



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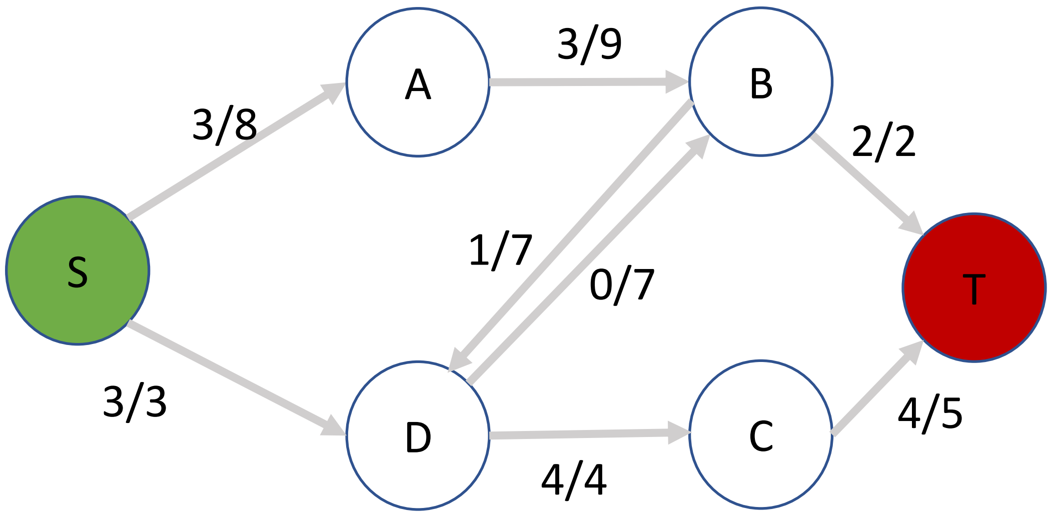
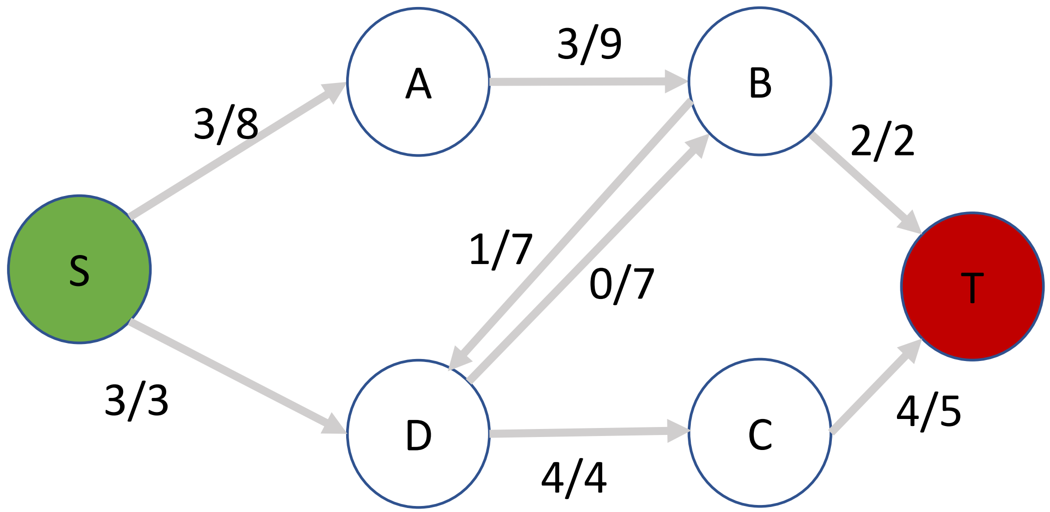
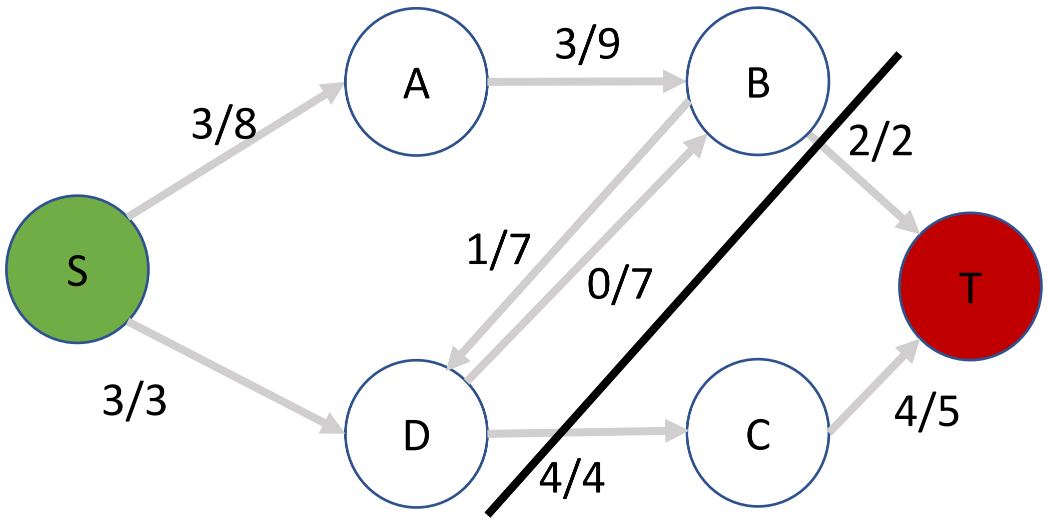
O(E*f)



From max flow to min cut



Theorem states that the maximum flow equals the total weight of the edges of a min cut



{S, A, B, D}
{C, T}
Min cut = 6

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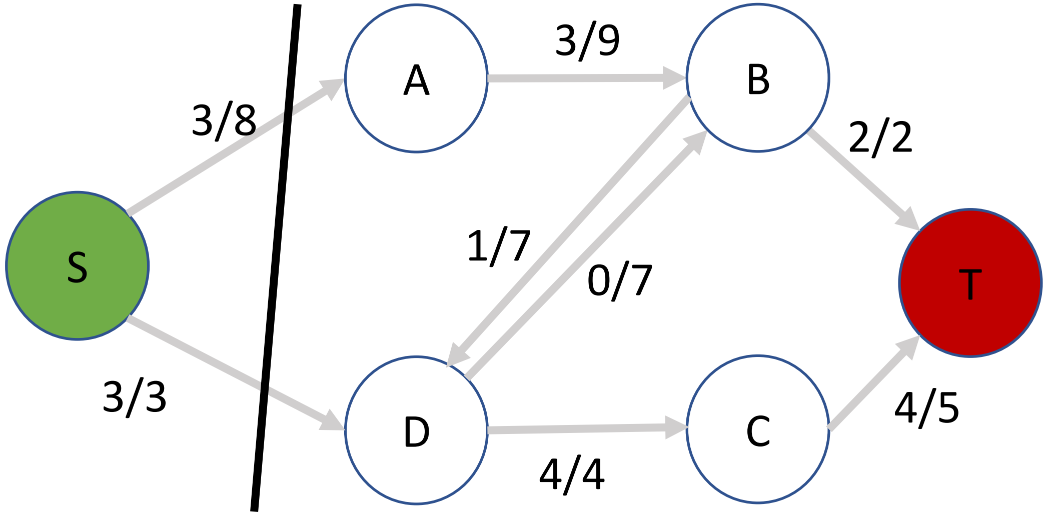

{S}
{A, B, C, D, T}
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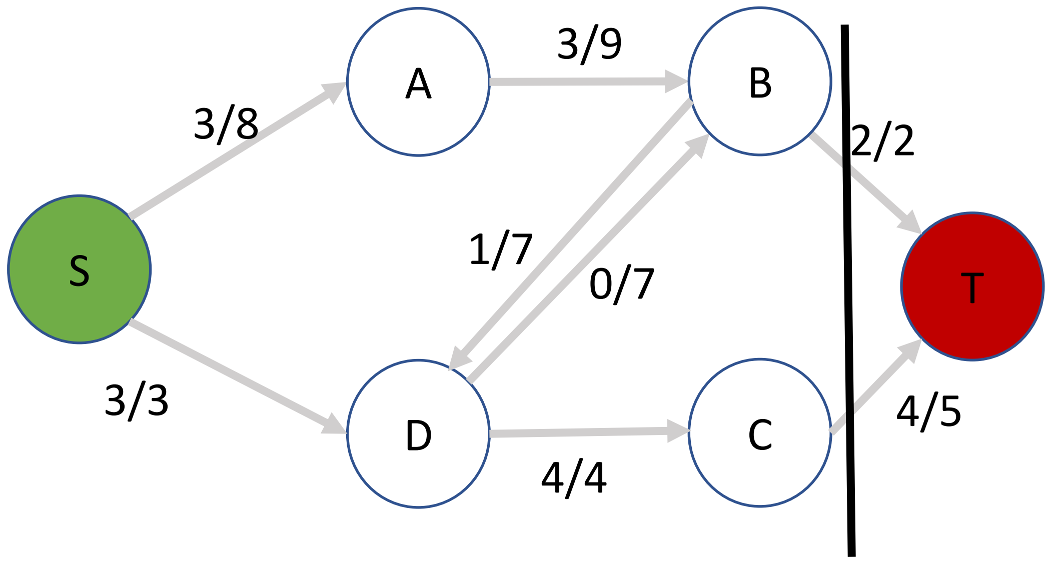

{A, B, C, D}
{T}
Min cut = 6

Assignment 6

Network Flow

Task 1 Flow Networks

a) List all the s-t **cuts in Flow Network Alpha and their capacities. Do this by describing the two sets of nodes the cut creates (For instance** $(\{s\}, \{x_1, x_2, t\})$**,** ***and*** **by drawing the cut graph(s).**

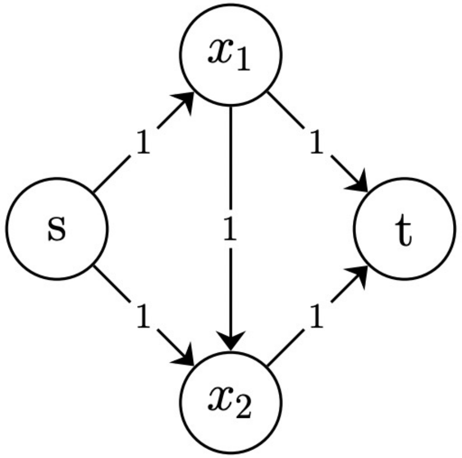

Figure 1: Flow Network Alpha

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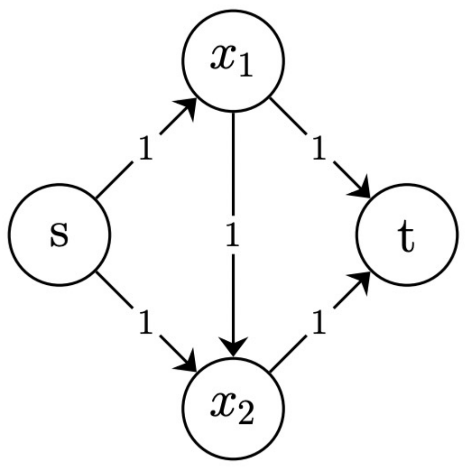

Figure 1: Flow Network Alpha

Create two sets A and B such that s is in A and t is in B, and that there is no overlap between the two sets

Set A: Nodes reachable from s after the cut
Set B: Nodes **not** reachable from s after the cut

Task 1 Flow Networks

a) **List all the s-t cuts in Flow Network Alpha and their capacities. Do this by describing the two sets of nodes the cut creates (For instance ($\{s\}$, $\{x_1, x_2, t\}$), *and* by drawing the cut graph(s).**

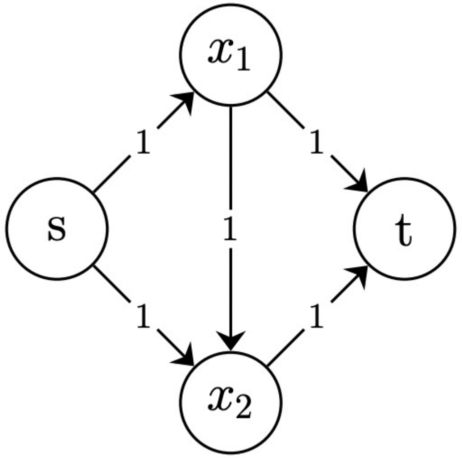

Figure 1: Flow Network Alpha

Set (A,B) such that there is no path from s to t.

Set A: Nodes reachable from s after the cut

Set B: Nodes **not** reachable from s after the cut

How to:

- Delete edges until you have such a set

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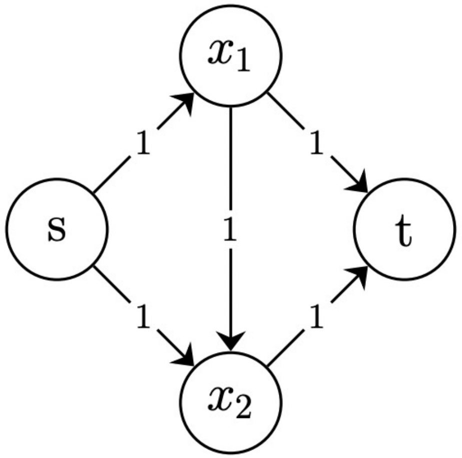
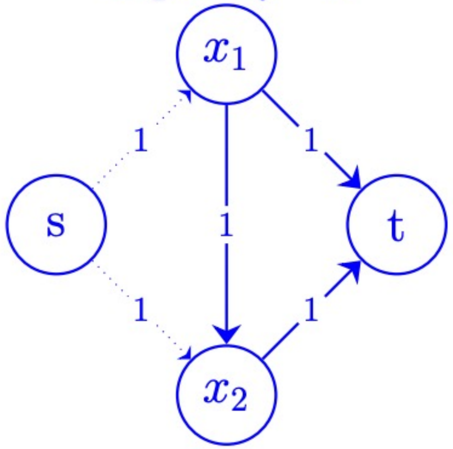


Figure 1: Flow Network Alpha

($\{s\}$, $\{x_1, x_2, t\}$)
capacity=2



Set (A,B) such that there is no path from s to t.

Set A: Nodes reachable from s after the cut
Set B: Nodes **not** reachable from s after the cut

How to:

- **Delete edges until you have such a set**
- **Capacity: Sum of the deleted edges**

Hint: There are four such cuts

c) What is the value of the computed flow in Flow Network Charlie? Is this a maximum (s, t) flow in this graph? Explain your answers.

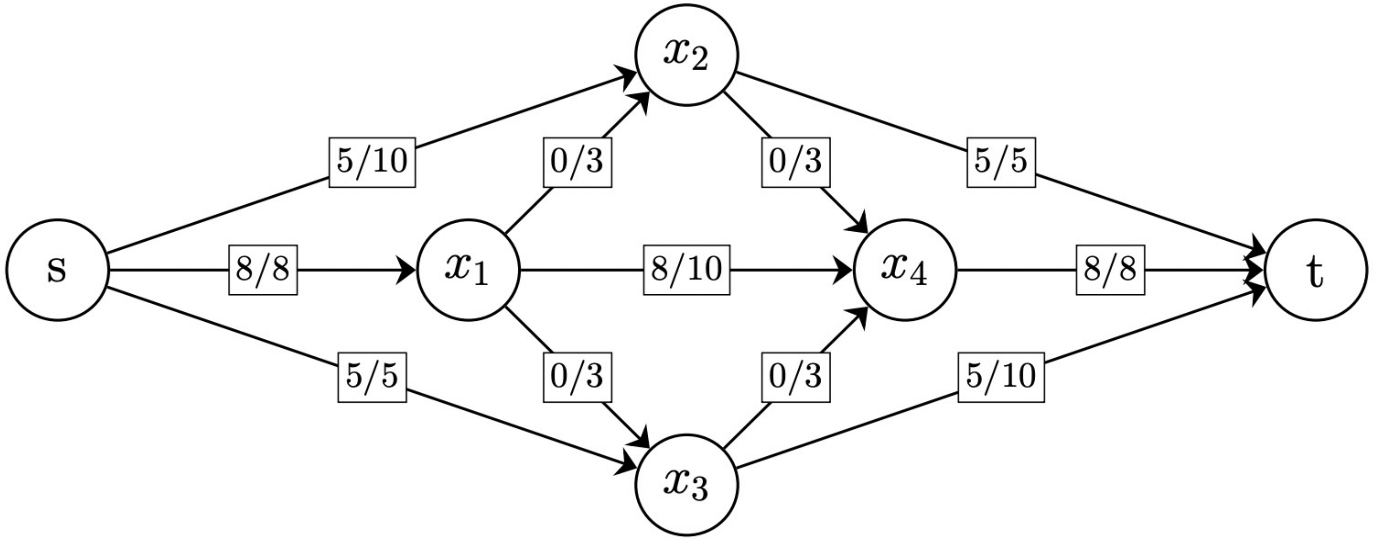


Figure 3: Flow Network Charlie. The flow and capacity of each edge appears as a boxed label on the edge in the form [*flow*/*capacity*].

Hint:

- Flow is the amount that leaves s and enters t
- Is this the maximum: Can more be sent using alternative routes?

- **What is the flow?**

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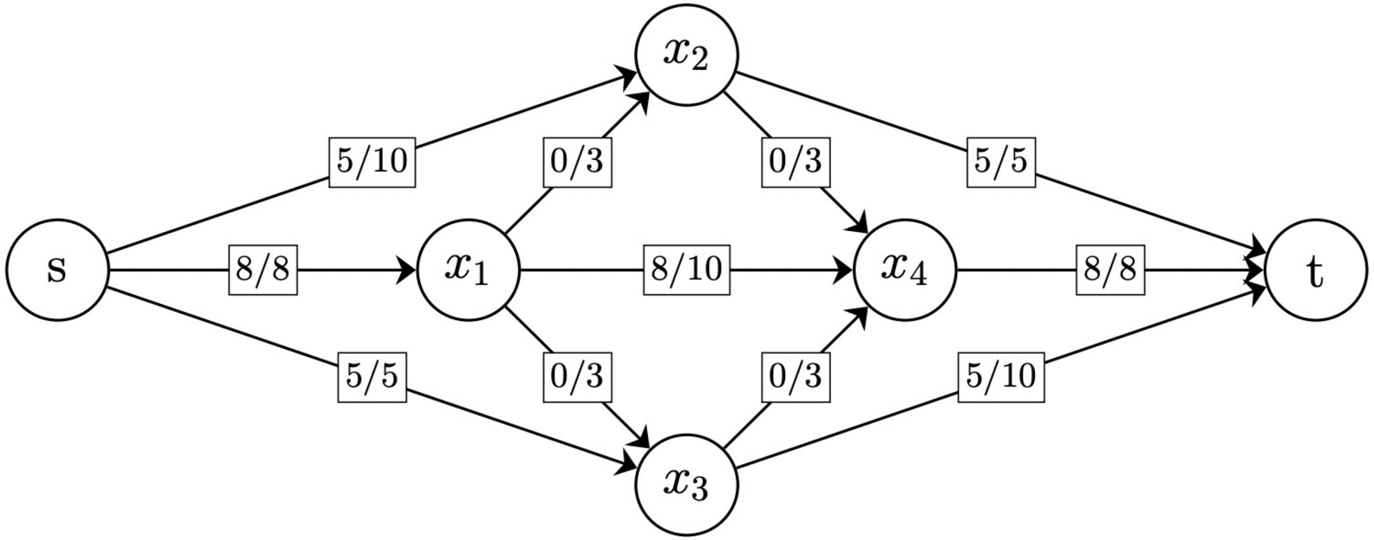

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Hint:

- Flow is the amount that leaves s and enters t
- Is this the maximum: Can more be sent using alternative routes?
- **What is the flow?**

If you know how to solve a) and c), you can solve b) and d) as well ;-)

e) Decide whether you think the following statements are true or false. If they are true, give a short explanation. If they are false, give a counterexample.

Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e.

i. If f is a maximum s-t flow in G, then f saturates every edge out of s with flow (i.e., for all edges e out of s, we have $f(e) = c_e$).

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For all such tasks: Draw a graph and try to contradict the statement

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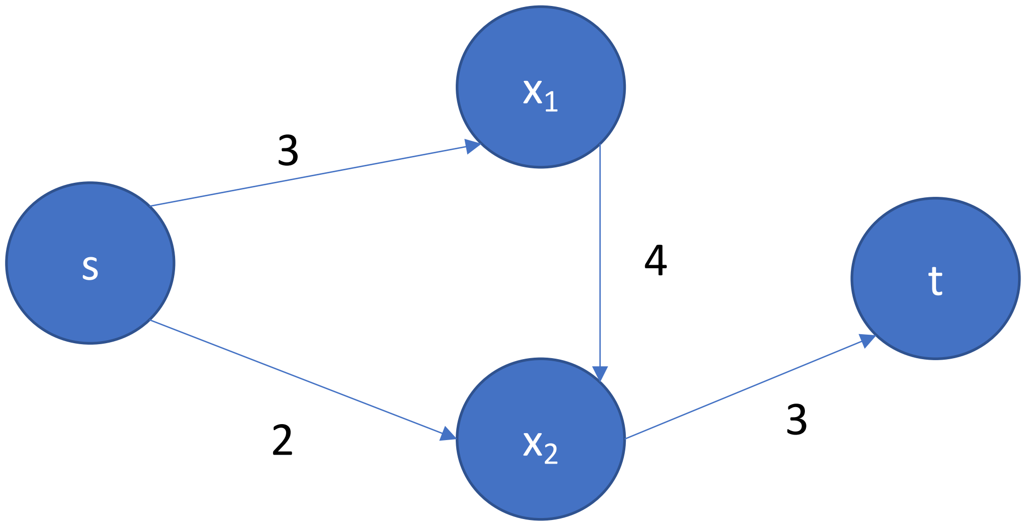
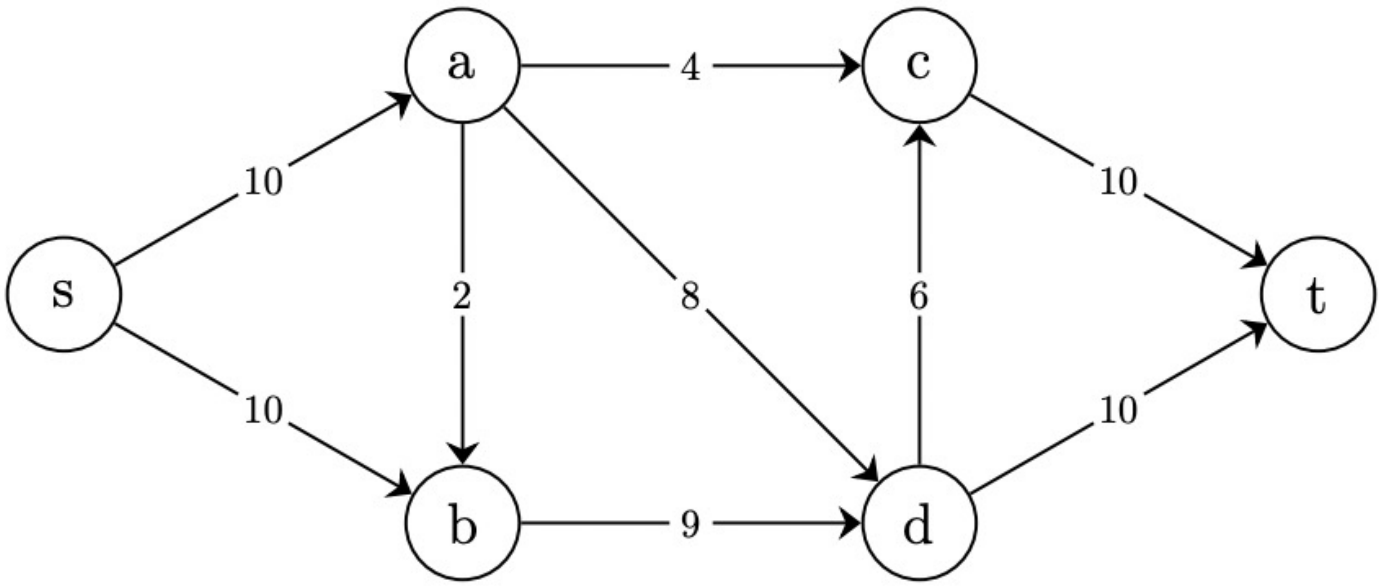


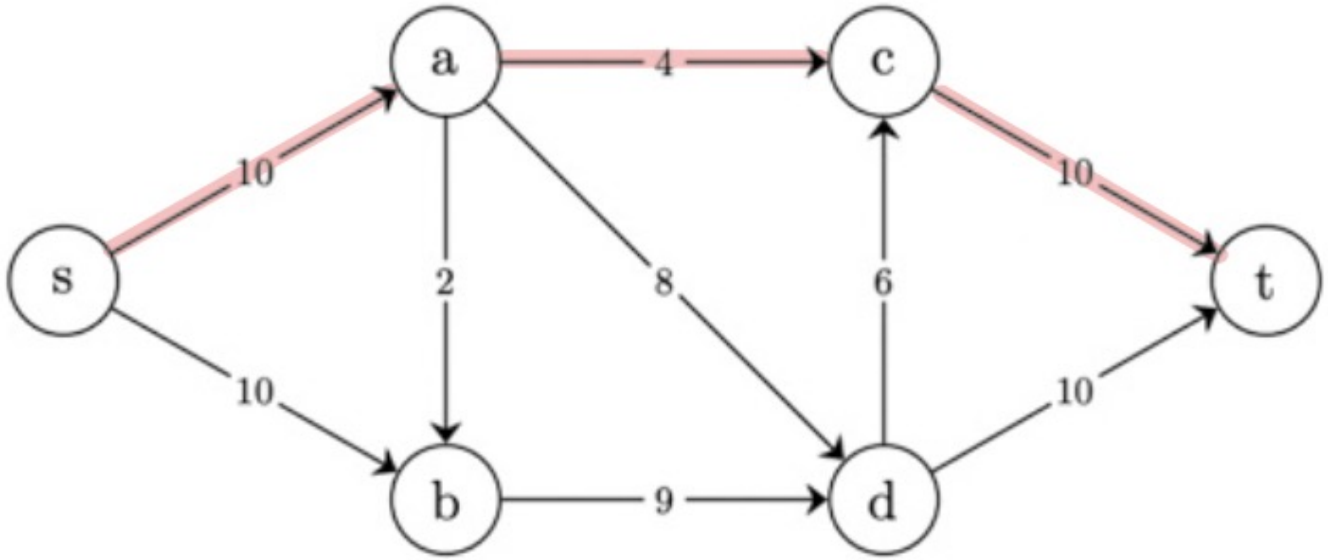
What is the maximum flow?

Task 2 Ford-Fulkerson

Given the following graph with capacities:

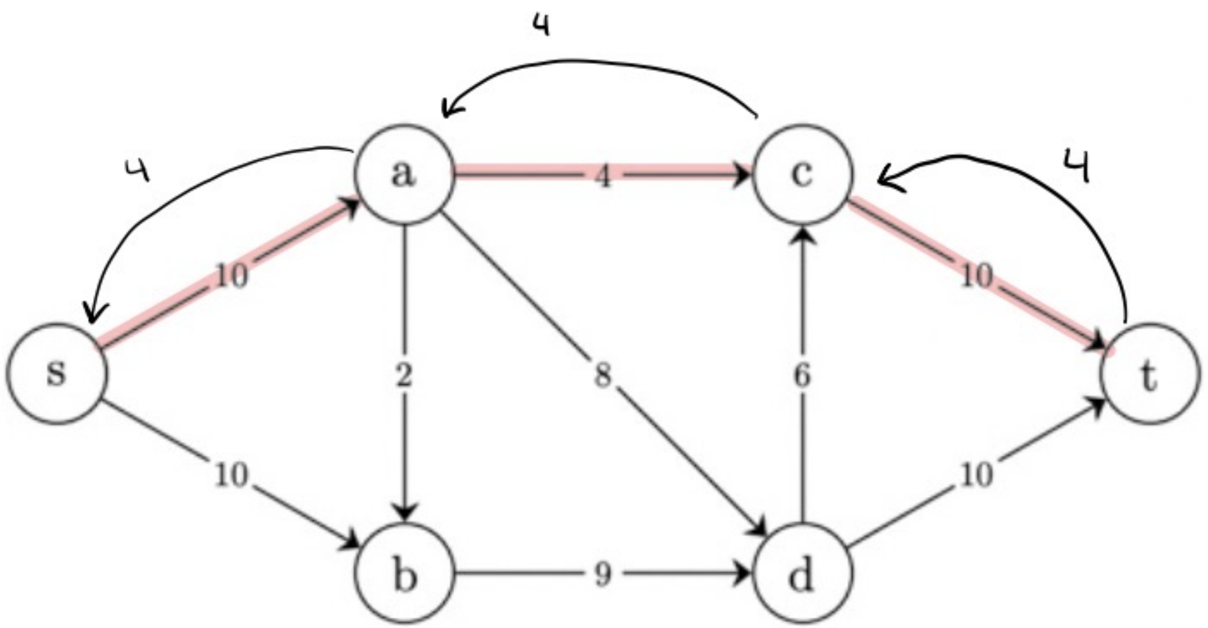


a) Apply the Ford-Fulkerson algorithm manually to find the maximum flow in this network. Show all augmenting paths. What is the Maximum Flow?



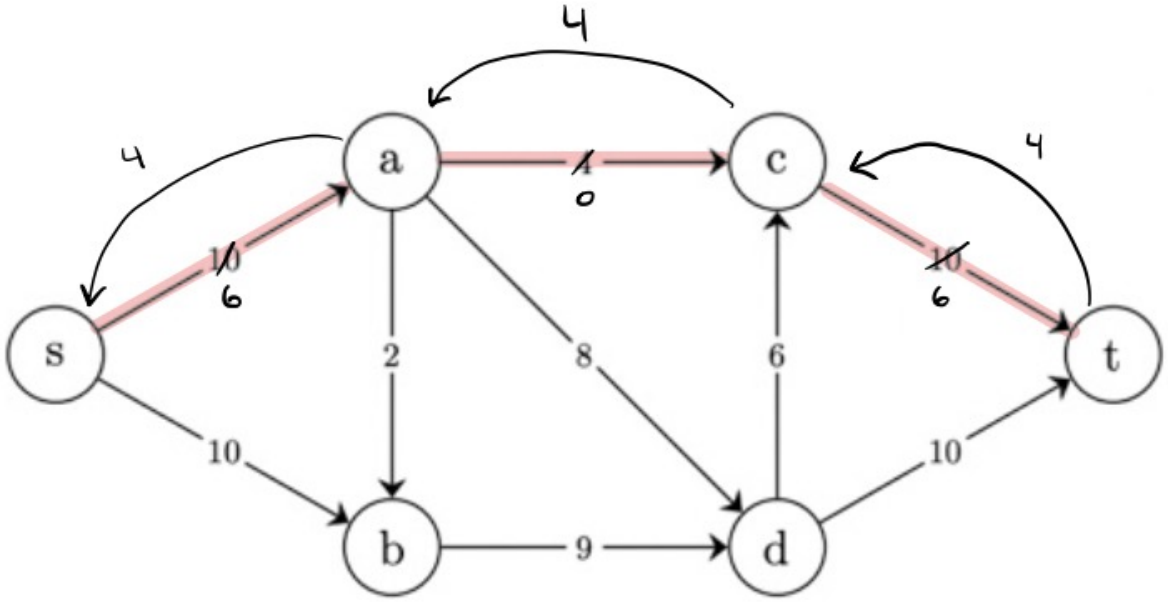
Step 1:

- **Find a path from s to t**
- **What is this paths maximum flow?**



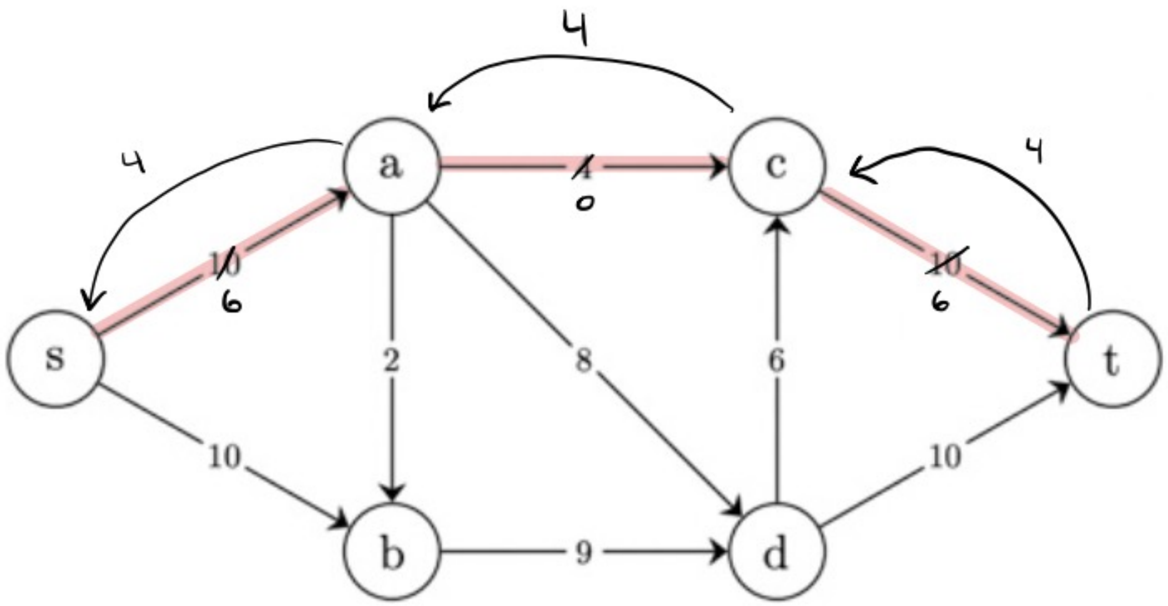
Step 2:

- **Send flow backwards**



Step 3:

- **Update capacities**



Step 3:
- **Update capacities**

Continue doing this until you have no more paths from s to t

Task 3 Blood Donors

Statistically, the arrival of spring typically results in increased accidents and increased need for emergency medical treatment, which often requires blood transfusions. Consider the problem faced by a hospital that is trying to evaluate whether its blood supply is sufficient.

The basic rule for blood donation is the following. A person's own blood supply has certain *antigens* present (we can think of antigens as a kind of molecular signature); and a person cannot receive blood with a particular antigen if their own blood does not have this antigen present. Concretely, this principle underpins the division of blood into four types: A, B, AB, and O. Blood of type A has the A antigen, blood of type B has the B antigen, blood of type AB has both, and blood of type O has neither. Thus, patients with type A can receive only blood types A or O in a transfusion, patients with type B can receive only B or O, patients with type O can receive only O, and patients with type AB can receive any of the four types.

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a)

Let s_O, s_A, s_B, and s_{AB} denote the supply in whole units of the different blood types on hand. Assume that the hospital knows the projected demand for each blood type d_O, d_A, d_B, and d_{AB} for the coming week. Explain how a flow algorithm can be used to evaluate if the blood on hand would suffice for the projected demand.

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Hints:

- The hospital has an overall supply and demand for blood
- You have donors and recievers (green)
- You have an constraint between these groups (yellow)

How can each of theese hints be translated to a network flow?

i. Is the 115 units of blood on hand enough to satisfy the 100 units of demand? Find an allocation that satisfies the maximum possible number of patients.

ii. Use an argument based on a minimum-capacity cut to show why not all patients can receive blood. Also, provide an explanation for this fact that would be understandable to the clinic administrators, who have not taken a course on algorithms. (So, for example, this explanation should not involve the words *flow*, *cut*, or *graph*).

Having solved the other parts of this assignment, you can solve this applying the same algorithms to your answer in a!
(There are also other ways to solve i) if you want to try)