NTNU

TDT4121 Introduction to Algorithms

Solution - Assignment 7

NP and Computational Intractability

Part 1 Theory

Task 1 NP and reduction

a) Explain in your own words what characterizes an NP-complete problem?

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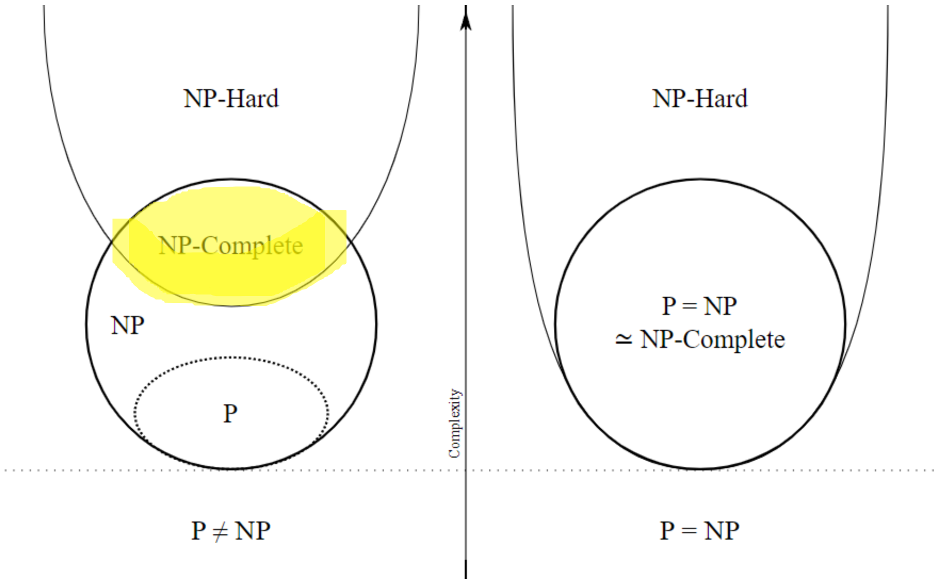

Part 1 Theory

Task 1 NP and reduction

a) Explain in your own words what characterizes an NP-complete problem?

Solution:
The book presents the following two properties an NP-complete problem posesses:

1. $X \in NP$
2. For all $Y \in NP, Y \leq_p X$

b) Your friend Steve has a problem A that he claims is NP-complete since he can prove that $A \in NP$. Is this sufficient proof that the problem is NP-complete? Explain why or why not.

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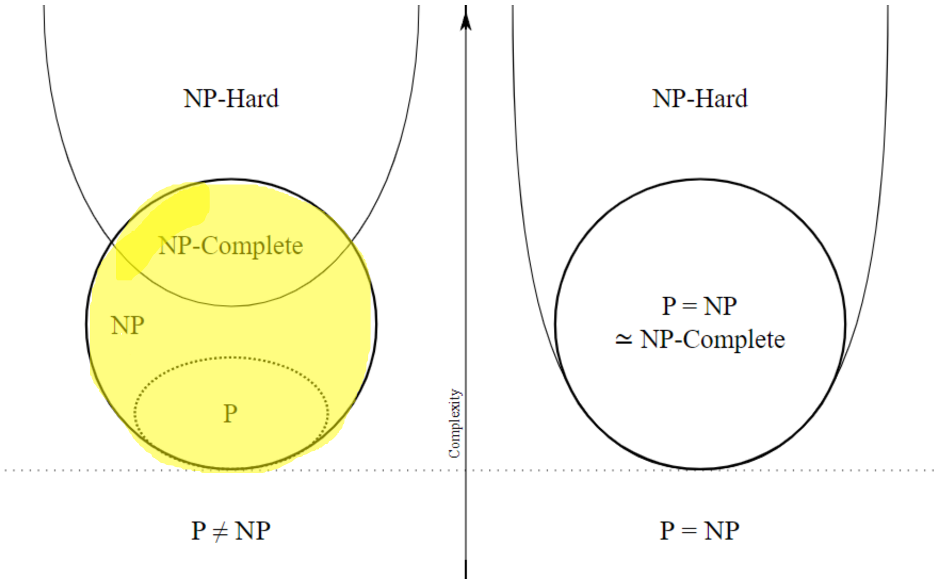

b) Your friend Steve has a problem A that he claims is NP-complete since he can prove that $A \in NP$. Is this sufficient proof that the problem is NP-complete? Explain why or why not.

Solution:

No, Steve has proved that the problem belongs to NP, but he also needs to prove that every other problem in NP is reducible to this problem. He could prove this by showing that a known NP-complete problem is reducible to his problem.

a) Recruitment at NTNU

The recruitment division at NTNU has the following problem. They need at least one professor skilled at each of the c courses at NTNU. They have received applications from p potential professors. For each of the c courses, there is some subset of the p professors qualified in that course. For a given number $k < p$, is it possible to hire at most k professors and have at least one qualified professor in each of the c courses?

We call this problem University Recruitment.

Show that Univerity Recruitment is NP-complete.

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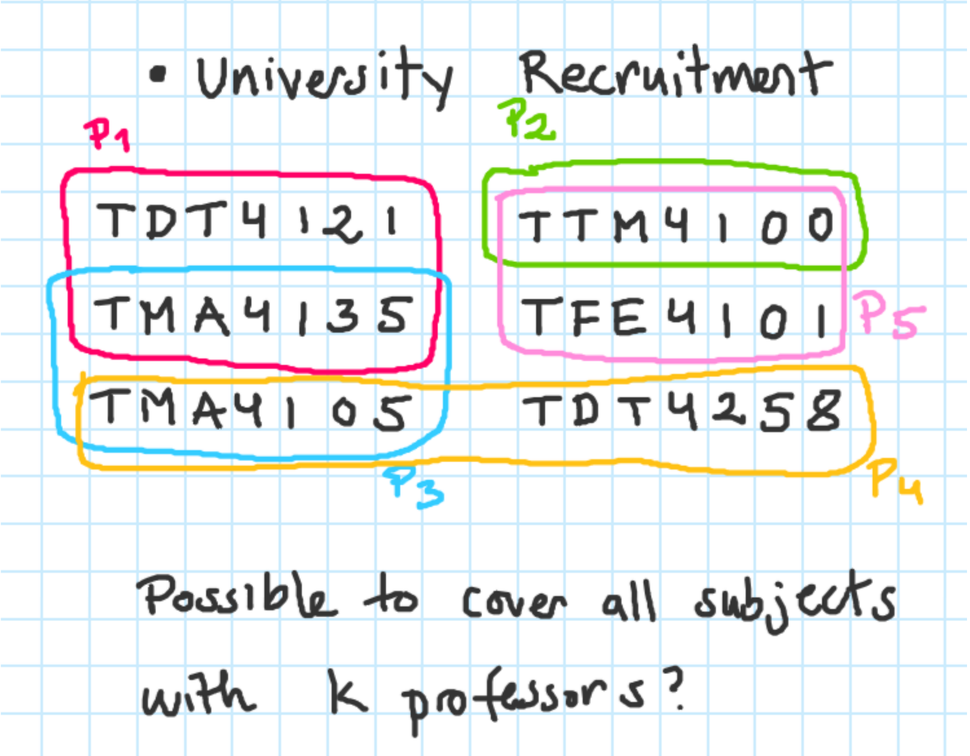

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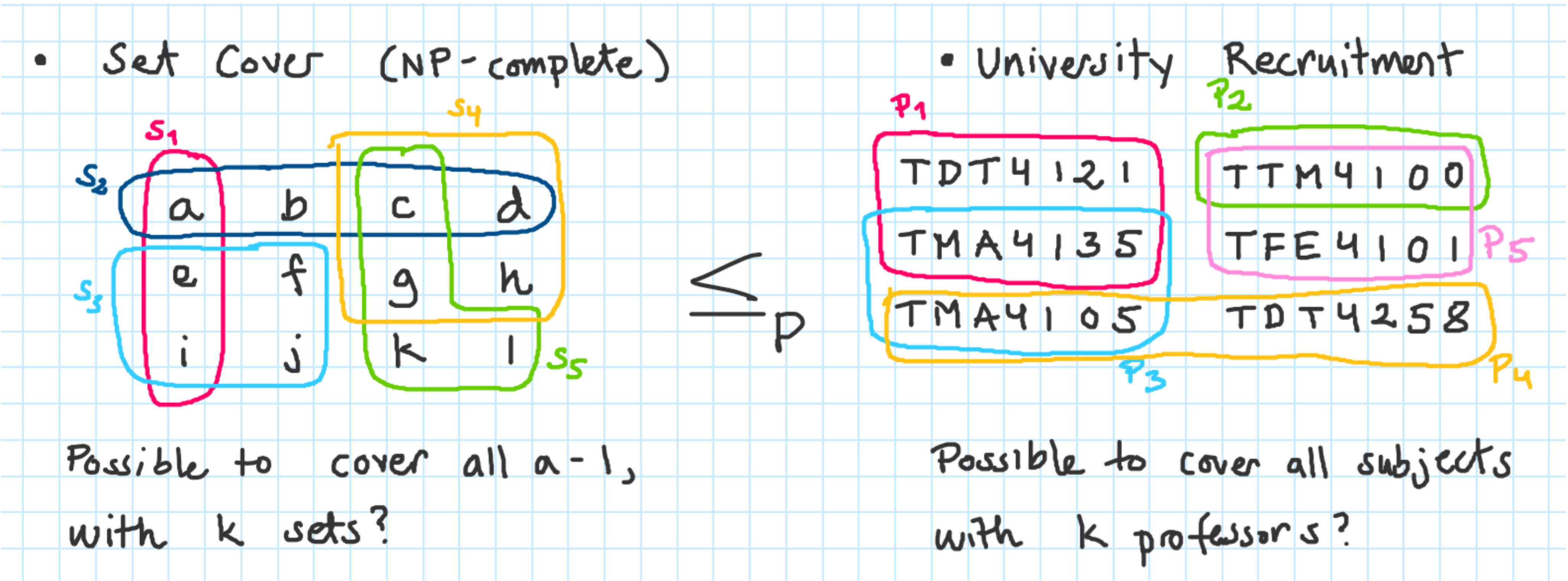

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Show that Univerity Recruitment is NP-complete.

Solution:
The problem is NP.

If we have a set of k professors, we can identify, in linear time ,the number of courses and k professors that is qualified. We will prove that this problem is NP-complete with a reduction from *set cover* problem. In the Set Cover problem we are given a set U consisting of n elements and a collection of m subsets whose union is the set of elements. The question is whether there are at most k of these sets whose union is equal to all of U. We can use Set cover to create an instance of University Recruitment: For each element of U create a course, for each of the m subsets create a professor and let this professor be qualified to teach the courses in the subset. This reduction takes polynomial time, and there are k professors that together are skilled in every sport, if there are k subsets, whose union is U.

Therefore we can say that Set Cover $\leq_p$ University Recruiting, meaning University Recruitment is an NP-complete problem.

b) Campus security

Campus security at NTNU has received a budget cut and has to let go of every security guard except for two: Nancy and Jonathan. They each go out in their car from the same starting location and return to the same location at the end of the day. They want to visit every building on campus. You want to ensure that the route they travel has the minimum possible overlap and that each location is covered by at least one car.

Let's call this problem Efficient Security.

Prove that this problem is NP-complete.

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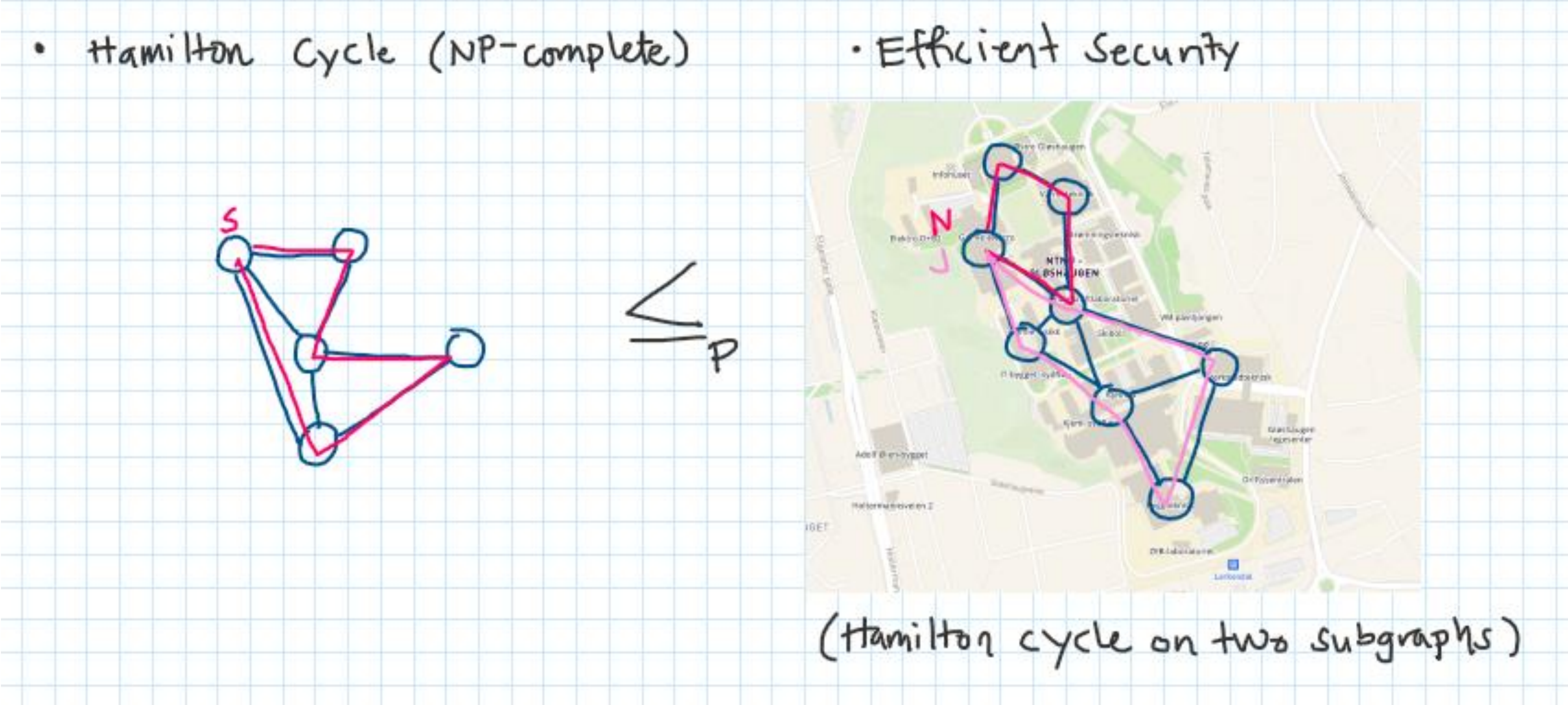

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Solution: /Hamilton cycle

If there is only one driver, then the Efficient Security problem reduces to the Traveling Salesman Problem (TSP), so this is in other words an extension of the travelling salesman problem which requires there to be two salesmen starting from the same position and fulfilling each others paths. The excact implementation of this is not that important, we know that we have to use the Traveling Salesman Problem, therefore we can say that Travelling Salesman $\leq_p$ Efficient Security

c) Campus parking

NTNU parking has received budget cuts, so now there is a very limited number of parking spots. Thankfully the staff at NTNU aren't all there at the same time. Meaning two professors can use the same parking spot at different times of the day. The head of parking at NTNU wants to make sure that the parking spots are scheduled so that no two professors park there at the same time. Each professor is only assigned to a single parking spot.

The problem is the following: Given a parking lot P, can we schedule the staff so that the parking spots do not overlap?

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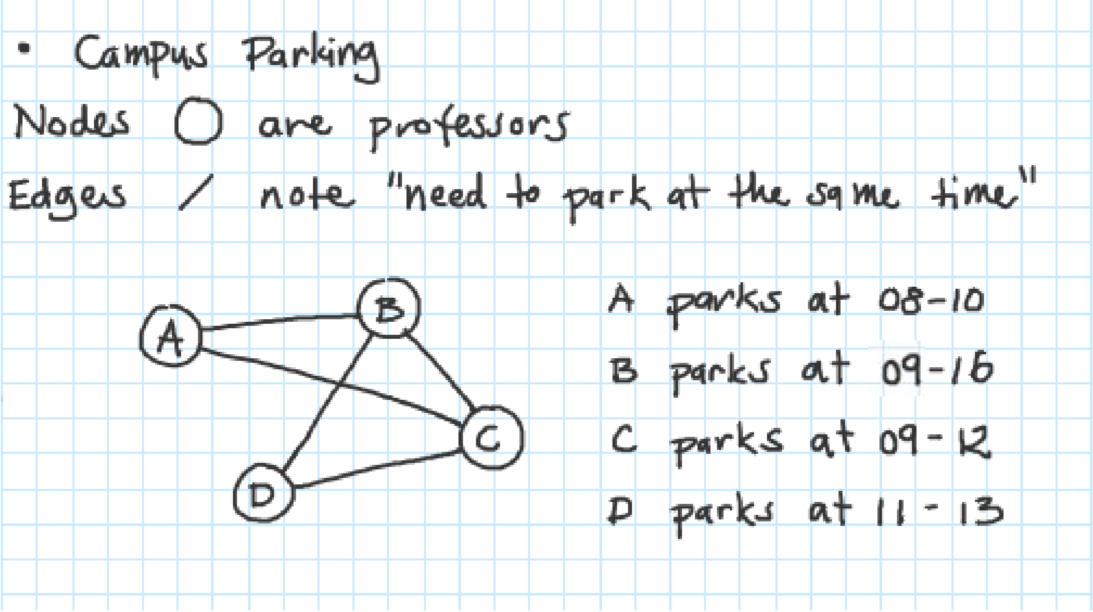

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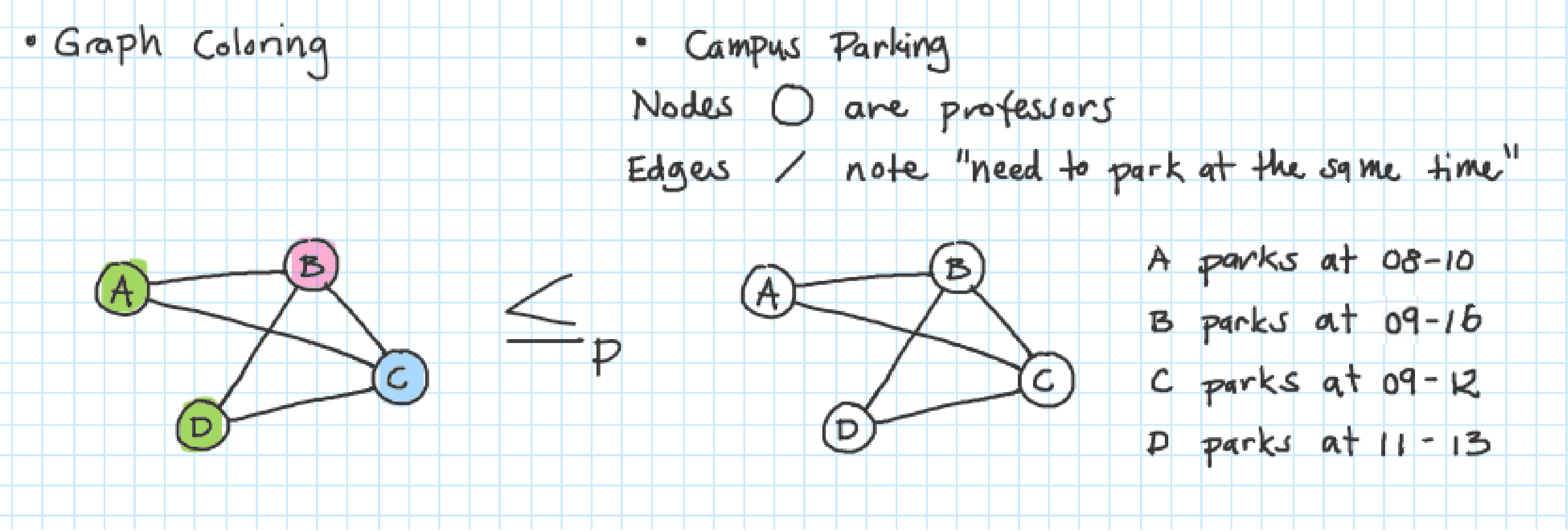

Task 3 The traveling salesman problem

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Solution:
This is an open question so there can be multiple correct answers, but some examples are:

- Many transportation and logistics applications
  - Public transport
  - Meal delivery
- Machine to drill holes in a certain sequence
- Circuit boards
- Computer wiring
- X-ray crystallography
- etc.

b) What is a Hamiltonian Cycle, and how is it used in the traveling salesman problem?

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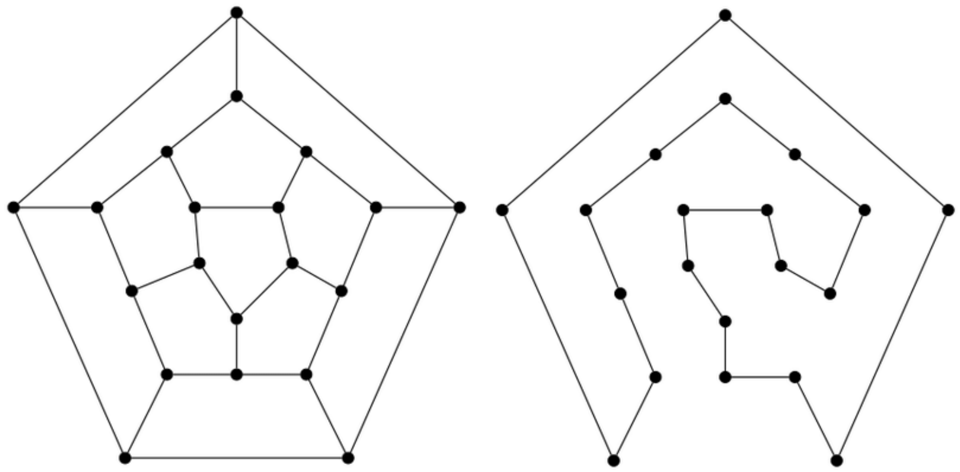
Solution:

A Hamiltonian cycle is a cycle in which all nodes (vertices) is only visited once.



(a) (b)

c) Given the following graph Gamma, give the shortest Hamiltonian cycle.

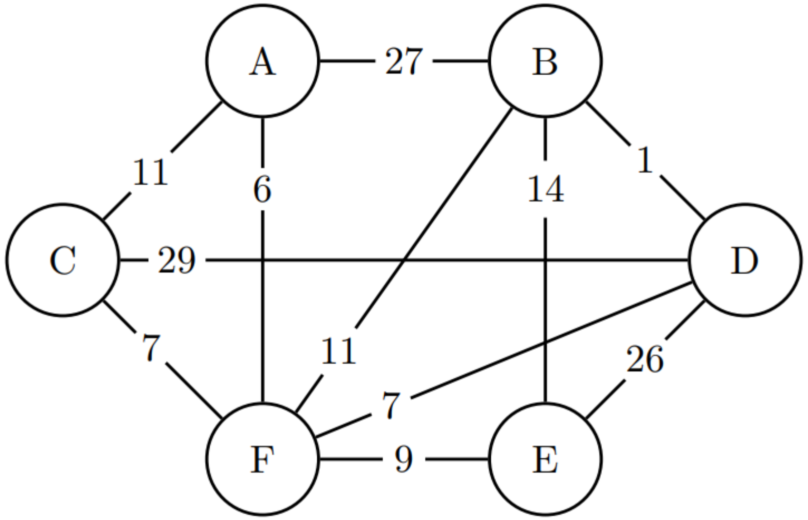


Figure 1: Gamma

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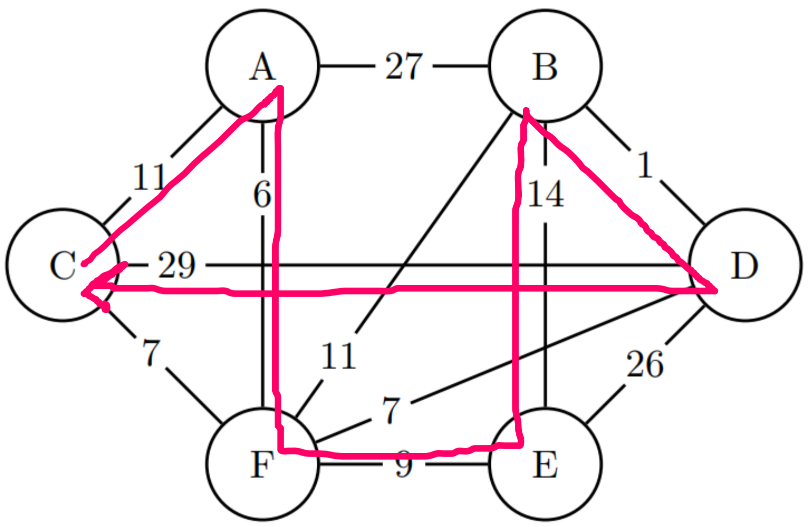

Figure 1: Gamma

Solution:
The shortest Hamiltonian cycle in this graph is C - A - F - E - B - D - C, which has a total weight of 52

Notice here that we used C as the starting vertice, but it doesn't matter where we start as it is a cycle.

Task 4 Partitoning problems

a) Explain what a perfect three-dimensional matching is

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Example problem instance:

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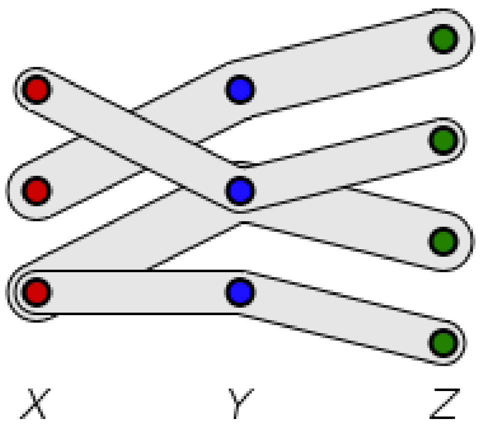

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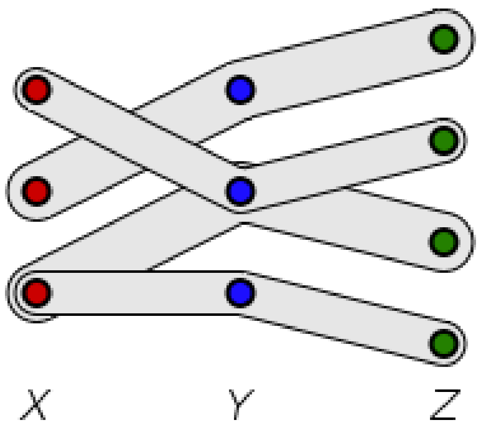
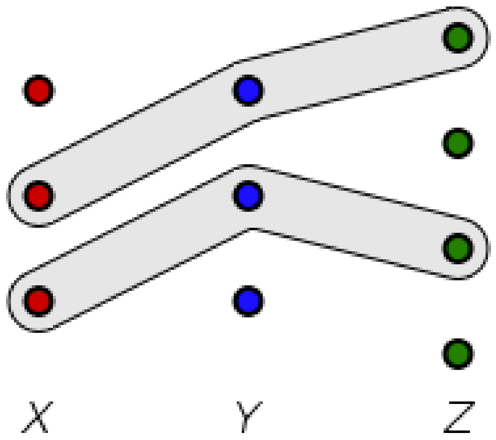


A solution:

(b)



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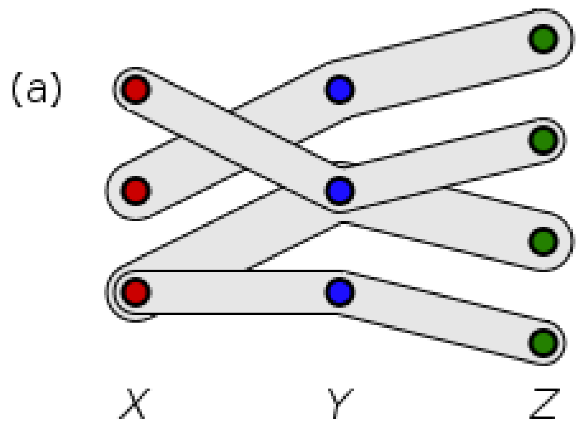

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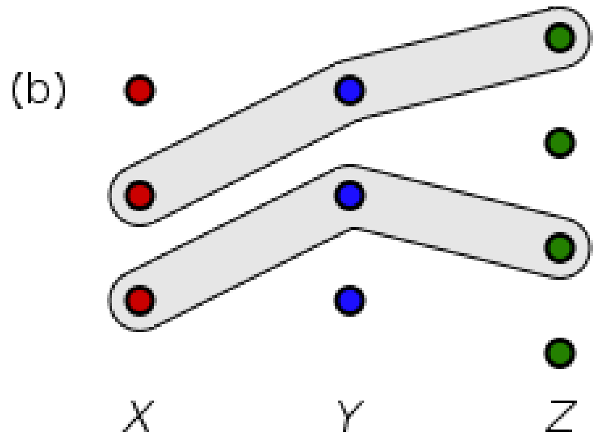
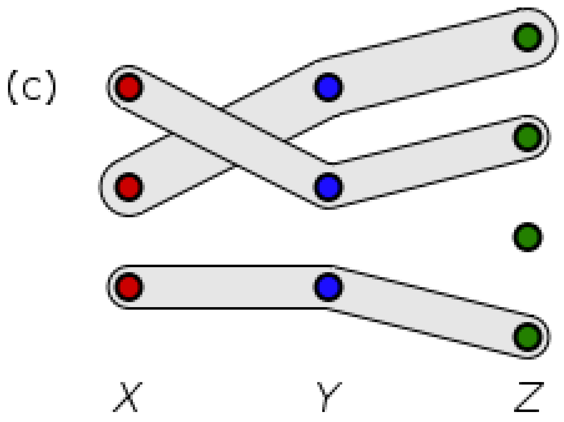
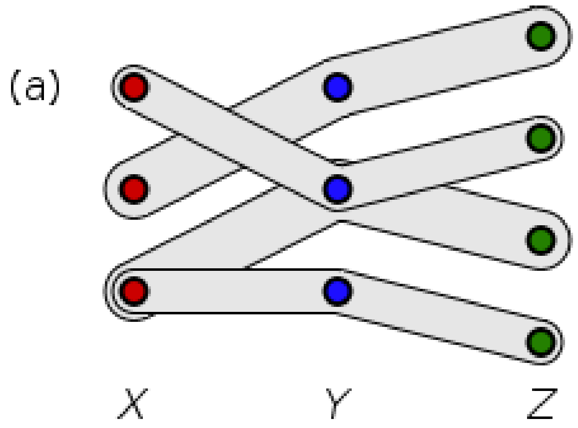


Another solution:





Task 4 Partitoning problems

a) Explain what a perfect three-dimensional matching is

Solution:

We have 3 finite sets X, Y, and Z, let T be a subset of $X \times Y \times Z$. Now $M \subset T$ is a three-dimensional matching if the following statement holds: for any two distinct triples $(x_1, y_1, z_1) \in M$ and $(x_2, y_2, z_2) \in M$, we have $x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2$

b) Is graph coloring with $k = 2$ NP-complete? Explain your answer

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Solution:

No, recall the bipartite graph problem from chapter 3. Coloring a graph with two colors is equivalent to that problem. We can solve the bipartite graph problem in polynomial time, therefore we can say that coloring a graph with two colors is **not** NP-complete.